MATH 233 LECTURE 32: VECTOR FIELDS

- These are vector-valued functions of several real variables. You should visualize a continuum of arrows in the plane (or in space). Mathematically, they are functions from $\mathbb{R}^2 \to \mathbb{R}^2$ (or $\mathbb{R}^3 \to \mathbb{R}^3$), and so may be considered as a pair (or triple) of the multivariable functions we have been studying. Notation: $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath}$ (or $\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$).
- An example you have already seen is the gradient of a function. In physics they arise as, for example, velocity fields (think of wind or another fluid) and force fields (gravitational, magnetic, electric).
- The *flow lines* of a velocity field $\vec{F}$ are the paths followed by a particle whose velocity at any point (x,y) is $\vec{F}(x,y)$. (For example, the flow lines of $\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$ are circles.) Parametrizing a flow line by $\langle x(t), y(t)\rangle$ leads to the equation $\langle x'(t), y'(t)\rangle = \vec{F}(x(t), y(t))$.
- A vector field $\vec{F}$ is called *conservative* if it is the gradient $\vec{\nabla} f$ of a function. Moving all the way around a closed curve (like a circle) against a conservative force field conserves energy, hence the terminology. Conservative fields are rather special: for example, $\vec{F}(x,y) = x\hat{\jmath}$ is not conservative. (Suppose $\vec{F} = \vec{\nabla} f$: can you find a contradiction?)