MATH 233 LECTURE 36: GREEN'S THEOREM IN THE PLANE

- Our first generalization of the Fundamental Theorem of Calculus related the line integral of a conservative vector field $\vec{F} = \vec{\nabla} f$ over a path (1-diml integral) to difference of the values of f at the endpoints ("0-diml integral"). As such, it is still a "1-dimensional" generalization. This lecture is about a bona fide *2-dimensional generalization*, relating a 2-dimensional integral to a 1-dimensional one.
- To state it, let C be a piecewise smooth closed curve (starts and ends at the same point), oriented in the counterclockwise direction, and enclosing a region S. (More precisely, we need that C is the boundary of; sometimes we write this as $C = \partial S$.) Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a continuously differentiable vector field on a region D containing C and S. Then Green's Theorem says that

$$\oint_C P dx + Q dy = \iint_S (Q_x - P_y) dA,$$

where $\oint$ is just standard notation for a line integral over a *closed* curve.
- Special case: if $\vec{F}$ is conservative, then $Q_x - P_y = 0$. So Green's theorem simply says that $\oint_C \vec{F} \cdot d\vec{r} = 0$, which we know from the last lecture.
- For a more interesting example, consider $\vec{F} = -\frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath}$. With this choice of P and Q, $Q_x - P_y = 1$, and Green's theorem says that $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \text{Area}(S)$.
- Warning: the vector field $\vec{F}$ must be defined on all of S for Green's Theorem to apply.
- I'll say more about why Green's theorem is true, and how to apply it when S has holes, in lecture.