# MATH 233 LECTURE 37: GREEN'S THEOREM; CURL AND DIVERGENCE

- First let's summarize what we have seen over the past few lectures. Let $\vec{F} = \langle P, Q \rangle$ be a vector field on a region $D \subset \mathbb{R}^2$. Then the following are equivalent:

  (a) $\int \vec{F} \cdot d\vec{r}$ is independent of path

  (b) $\oint \vec{F} \cdot d\vec{r} = 0$ on all closed curves

  (c) $\vec{F}$ is conservative (i.e. is the gradient field of some function $f(x, y)$ on $D$)

  Moreover, if any (hence all) of (a), (b), (c) hold, then we have

  (d) $Q_x - P_y$ is identically zero.

  This only implies the other three (hence is equivalent to them) if $D$ is simply connected.
- We will give some examples of how to use Green's theorem to compute work and area.
- Now let $\vec{F} = \langle P, Q, R \rangle$ be a vector field on a region $D$ in $\mathbb{R}^3$. Define the *divergence* of $\vec{F}$ by

$$\text{div}(\vec{F}) := \vec{\nabla} \cdot \vec{F} := P_x + Q_y + R_z$$

  and the *curl* of $\vec{F}$ by

$$\text{curl}(\vec{F}) := \vec{\nabla} \times \vec{F} := \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

  Notice that $\text{div}(\vec{F})$ is a function whereas $\text{curl}(\vec{F})$ is a vector field.
- We can define div and curl in 2 dimensions too: if $\vec{F} = \langle P, Q \rangle$, you can think of this as a 3-D vector field with $R = 0$ and $P$ and $Q$ constant in the $z$-direction. Thendiv$(\vec{F}) = P_x + Q_y$ while $\text{curl}(\vec{F}) = (Q_x - P_y)\hat{k}$. (So for all practical purposes curl is a *function* in the 2-D setting.)

- If we think of $\vec{F}$ (in 3-D or 2-D) as measuring the velocity field of a fluid, then intuitively $\mathrm{div}(\vec{F})(p)$ (we are evaluating the function at $p$ to get a number) measures the tendency of the fluid to diverge away from $p$ (or accumulate toward it, if negative). [I'll quantify this in class using a notion called "flux".] We say $\vec{F}$ is *incompressible* if $\mathrm{div}(\vec{F})$ is identically zero. On the other hand, if the fluid rotates about an axis at $p$, then $\mathrm{curl}(\vec{F})$ points along the axis in the direction given by the "right-hand rule". If the curl is zero everywhere we say $\vec{F}$ is *irrotational*.