MATH 233 LECTURE 38: GAUSS AND STOKES THEOREMS IN THE PLANE

More on curl and div.

- Though the title says "plane", we'll start with some stuff in space. Recall that for a vector field $\vec{F}$ on a region $D \subset \mathbb{R}^3$, $\text{div}\vec{F} = \vec{\nabla} \cdot \vec{F}$ and $\text{curl}\vec{F} = \vec{\nabla} \times \vec{F}$, where $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$.
- By Clairaut's Theorem (details in class), we have two identities: (1) $\text{curl}(\vec{\nabla} f) = \vec{0}$ for any function f on D; and (2) $\text{div}(\text{curl}\vec{F}) = 0$ for any vector field $\vec{F}$ on D.
- By (1), if $\vec{F} = \vec{\nabla} f$ (i.e. $\vec{F}$ conservative), then $\text{curl}\vec{F} = \vec{0}$ (i.e. $\vec{F}$ irrotational). If D is simply connected, then the converse holds: $\vec{F}$ irrotational $\implies$ $\vec{F}$ conservative.
- By (2), if $\vec{F}$ is the curl of another vector field $\vec{G}$ (i.e. $\vec{F}$ is a "curl field"), then $\text{div}\vec{F} = 0$ (i.e. $\vec{F}$ incompressible). If D has no "solid holes", then the converse holds here too.

Vector forms of Green's Theorem.

- Let C be a simple closed curve in $\mathbb{R}^2$, with a smooth parametrization $\vec{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath}$ by arclength s, and "positively oriented" (i.e. in the counter-clockwise direction). The unit tangent vector is $\hat{T}(s) = x'(s)\hat{\imath} + y'(s)\hat{\jmath}$, and the outward-pointing unit normal is $\hat{n}(s) = y'(s)\hat{\imath} - x'(s)\hat{\jmath}$.
- Now suppose $C = \partial S$, and that $D \subset \mathbb{R}^2$ contains C and S. Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on D (D contains C). Write

$$\oint_C \vec{F} \cdot \hat{n} ds = \oint_C (P\hat{\imath} + Q\hat{\jmath}) \cdot (y'(s)\hat{\imath} - x'(s)\hat{\jmath}) ds$$

$$= \oint_C -Qx'(s)ds + Py'(s)ds = \oint_C -Qdx + Pdy,$$

which by Green's Theorem

$$= \iint_S (P_x - (-Q_y))dA = \iint_S (P_x + Q_y)dA.$$

This gives Gauss's Divergence Theorem in the plane:

$$\oint_{\partial S} \vec{F} \cdot \hat{n} ds = \iint_S \mathrm{div}(\vec{F}) dA,$$

which tells us that the total flux of $\vec{F}$ across the boundary ∂S (the left-hand side) equals the integral of the "outward flux per unit area" over S, which sounds completely plausible.

- There is also "Stokes's Theorem in the plane" which is more or less a restatement of Green's theorem: it reads

$$\oint_{\partial S} \vec{F} \cdot \hat{T} ds = \iint_S (\mathrm{curl}\vec{F}) \cdot \hat{k} dA.$$

Harmonic functions and Maxwell's equations.

- The Laplacian is the operator $\nabla^2 := \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. It may be applied to functions or vector fields. Notice that $\nabla^2 f = \mathrm{div}(\vec{\nabla} f)$.
- We say that f is harmonic if $\nabla^2 f = 0$, and similarly for vector fields.
- Let $\vec{E}(x, y, z; t)$, $\vec{H}(x, y, z; t)$ denote the electric and magnetic fields (vector fields in space that change in time t). The simplest presentation of Maxwell's equations (in a vacuum) is:

$$\vec{\nabla} \cdot \vec{E} = 0 = \vec{\nabla} \cdot \vec{H}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

where c is the speed of light. (The units are not natural in this form and I won't address them here.) In class, I will say how to derive the wave equations

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} , \qquad \nabla^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

from them. Notice that this says that, for example, $\vec{E}$ *is static (doesn't change with time) if and only if it is harmonic.*