MATH 233 LECTURE 9 (§13.3): ARC-LENGTH AND CURVATURE

As usual, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a vector valued function, with derivative $\vec{r}'(t)$ and speed $\|\vec{r}'(t)\|$, tracing out a curve C as t varies from a to b.

Arclength.

- On an infinitesimal level, the change in arclength is given by the speed times the change in time.
- This leads to the integral formula for the arclength of C: $s = \int_a^b \|\vec{r}'(t)\|\, dt$.

Curvature.

- This is a measure of how tightly curved C is (which can vary from point to point), or more precisely, of the rate of change of the direction of the tangent vector with respect to arclength. (We want the curvature of C at a point to be independent of the choice of parametrization $\vec{r}(t)$; this is why we measure with respect to arclength and not time.)
- One way to compute it: parametrize C by arclength, i.e. by an $\vec{r}(t)$ with unit speed ($\|\vec{r}'(t)\| = 1$ everywhere). In this case, arclength $s = t$ and curvature $\kappa(s) := \|\vec{r}''(s)\|$. (Warning: don't use this formula unless you have a parametrization by arclength!)
- General formula: $\kappa(t) = \left\|\vec{T}'(t)\right\| / \|\vec{r}'(t)\|$, where $\vec{T}(t) = \vec{r}'(t) / \|\vec{r}'(t)\|$. (This gives the curvature at the point $\vec{r}(t)$ on C.)
- The curvature at any point on a circle is the reciprocal of the circle's radius. Small circle, big curvature; big circle, small curvature.