Lecture 1 : Linear Systems

Geometric viewpoint

For simplicity, start with systems of n linear equations in n unknowns, where $n = 2$ or 3.

These will have 3 presentations :

(a) by "row" equations

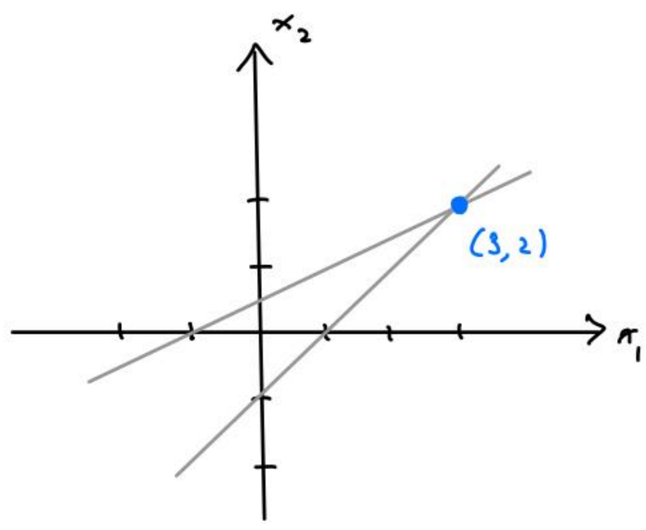
(b) by "column" equation

(c) by "matrix" equation.

Ex 1/ Consider the system (in form (a))

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + x_2 = -1 \end{cases},$$

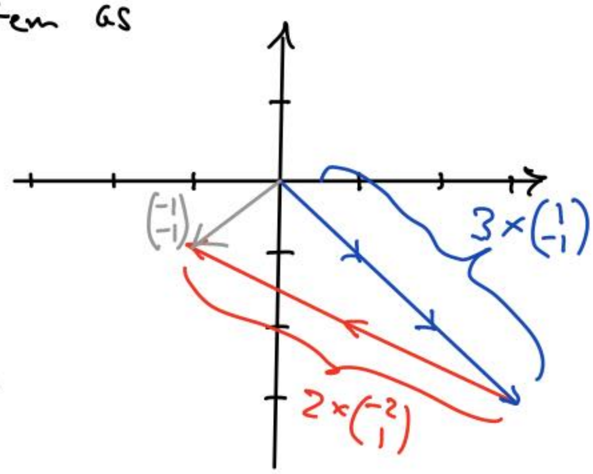
with accompanying picture :



... from which we see that $(x_1, x_2) = (3, 2)$ is the unique solution of the system.

(b) Now we can rewrite the system as

$$x_1\begin{pmatrix}1\\-1\end{pmatrix} + x_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$



which asks the question:

"Can we produce $\begin{pmatrix}-1\\-1\end{pmatrix}$ as a <u>linear</u> <u>combination</u> of $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\1\end{pmatrix}$?"

The answer, as shown in the picture, is YES.

(c) The matrix form of the equation is

$$\begin{pmatrix}1 & -2\\-1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\text{“}\ A \cdot \vec{x} = \vec{b}\ \text{”}$$

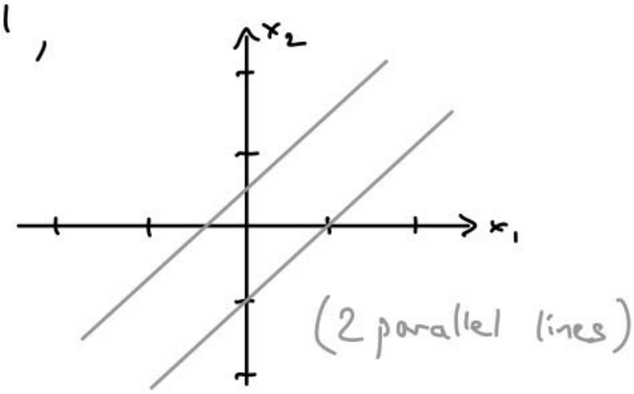
//

The system in the Example is called <u>consistent</u> because a solution exists. Here is an <u>inconsistent</u> one:

Ex 2/ If we change the first equation in Ex. 1 to

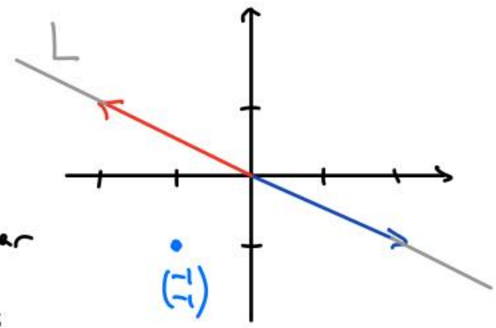
$$2x_1 - 2x_2 = -1,$$

then picture **(a)** becomes



While in **(b)** we have

$$x_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$



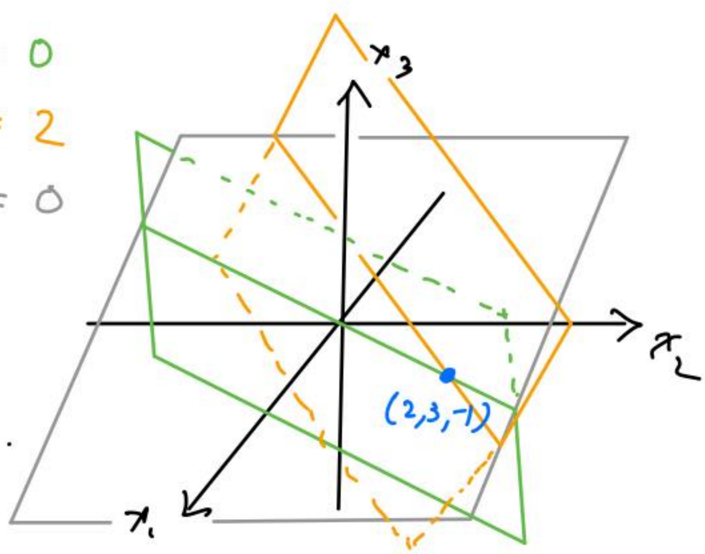
which is impossible (as any linear combination of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ lies on the line L, and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ does not). //

Ex 3/ Finally, if we change $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ to something lying on the line L, say $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$, then (in **(b)**) there are <u>many</u> linear combinations of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ that will do. Correspondingly, the two parallel lines in picture **(a)** of Example 2 <u>merge</u>, and we have <u>infinitely many solutions</u> ($\Rightarrow$ consistent). //

Turning to $n = 3$, here is a consistent example:

Ex 4/ **(a)** $\boxed{\text{ROW}}$

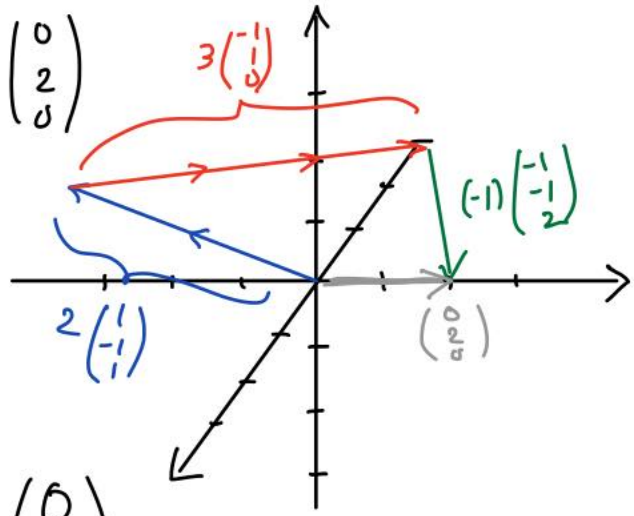
3 planes in space
$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_3 = 2 \\ x_1 \qquad + 2x_3 = 0 \end{cases}$$



Note that two of the planes pass through the origin (0,0,0). (why?)

$\boxed{\text{COLUMN}}$

(b) $x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$



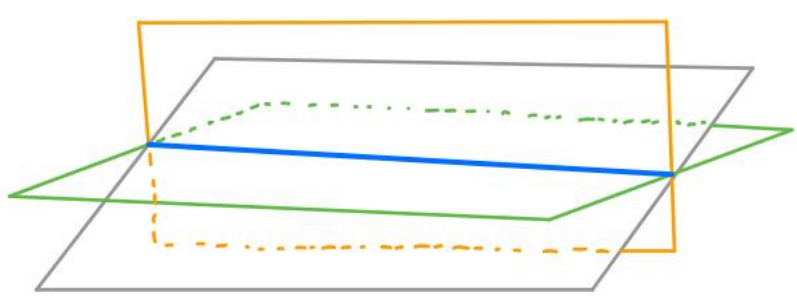
$\boxed{\text{MATRIX}}$

(c) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

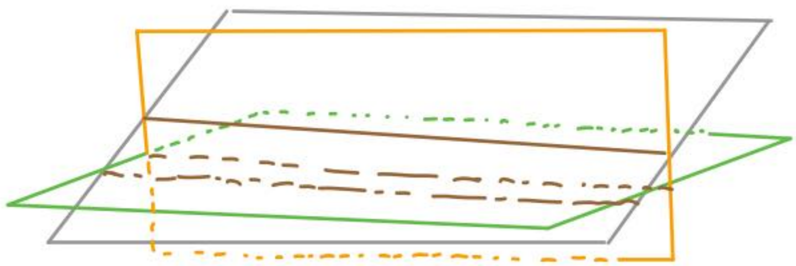
Ex 5/ If in Ex. 4(a), we made the last equation

$$x_3 = -1,$$

we would get a line as solution set:



On the other hand, if we move the $x_3 = -1$ plane up or down to $x_3 = a \ (\neq -1)$, then we get the picture



so that there are no common solutions (and the system is inconsistent).

Correspondingly, what happens with the vector equations? The linear combinations

$$x_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + x_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix} \quad \text{include } \begin{pmatrix}0\\2\\-1\end{pmatrix}, \text{ but } \underline{\underline{\text{NOT}}} \begin{pmatrix}0\\2\\a\end{pmatrix} \text{ for } a \neq -1.$$

From these examples we can (informally) glean that

(i) There are 3 possibilities for linear systems: no solutions, one solution, or infinitely many.

(ii) The n equations have a common solution (i.e. are consistent) $\iff$ the column vector $\vec{b}$ on the right-hand-side of the vector equation can be written as a linear combination of the column vectors on the left-hand-side.

(iii) The equations have a common solution for every $\vec{b}$ $\iff$ linear combinations of the left-hand-side column vectors fill up all of n-space.

Algebraic viewpoint

This will be a first glimpse of Gaussian elimination/row operations, to be made more systematic in subsequent lectures.

Ex 6 / Find the solution set (possibly empty) of the system

$$\left.\begin{array}{ll} (\rho_1) & x_1 + x_2 + x_3 = 9 \\ (\rho_2) & 2x_1 + 4x_2 - 3x_3 = 1 \\ (\rho_3) & 3x_1 + 6x_2 - 5x_3 = 0 \end{array}\right\} \quad \xrightarrow{\text{Shorthand}} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

"AUGMENTED MATRIX"

eliminate x_1 in last 2 equations $\Big\downarrow$ $\rho_2 \mapsto \rho_2 - 2\rho_1$, $\rho_3 \mapsto \rho_3 - 3\rho_1$

$$\left.\begin{array}{ll} (\rho_1) & x_1 + x_2 + x_3 = 9 \\ \text{new } (\rho_2) & 2x_2 - 5x_3 = -17 \\ \text{new } (\rho_3) & 3x_2 - 8x_3 = -27 \end{array}\right\} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right]$$

eliminate x_2 in last equation $\Big\downarrow$ $\rho_3 \mapsto \rho_3 - \frac{3}{2}\rho_2$

$$\left.\begin{array}{ll} (\rho_1) & x_1 + x_2 + x_3 = 9 \\ (\rho_2) & 2x_2 - 5x_3 = -17 \\ \text{new } (\rho_3) & -\frac{1}{2}x_3 = -\frac{3}{2} \end{array}\right\} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

So $x_3 = 3$, and back-substituting in (ρ_2) gives $2x_2 - 15 = 17$ $\Rightarrow x_2 = -1$, whereupon substituting in (ρ_1) gives $x_1 - 1 + 3 = 9$ $\Rightarrow x_1 = 7$. You can check that $(7, -1, 3)$ solves the original system. //

Why does this work? Applying

Elementary Row Operations

(a) **Replace** an equation/row by $\left\{ \text{itself} + \text{multiples of other equations/rows} \right\}$

(b) **Swap** two equations/rows

(c) **Scale** an equation/row (multiply by a nonzero constant)

produces a new "row-equivalent" system of equations whose solution set certainly includes all the old solutions. In fact, since (a)-(c) are reversible, the new solution set is the same:

Row-equivalent systems are equivalent.

Here's an example that uses all 3 operations:

Ex 7/

$$\left.\begin{aligned} x_3 - x_4 &= -1 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 &= 2 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 &= 3 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 &= 6 \end{aligned}\right\} \to \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{(b):\ \rho_1 \leftrightarrow \rho_2} \left[\begin{array}{cccc|c} 2 & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right] \xrightarrow{(c):\ \rho_1 \mapsto \frac{1}{2}\rho_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right] \xrightarrow[\substack{\rho_3 \mapsto \rho_3 - 2\rho_1 \\ \rho_4 \mapsto \rho_4 - 3\rho_1}]{(a):}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{array}\right] \xrightarrow[\substack{\rho_3 \mapsto \rho_3 - \rho_2 \\ \rho_4 \mapsto \rho_4 - 3\rho_2}]{(a):} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

STOP. The 3rd line corresponds to the equation $0 = 2$, and so the system is inconsistent (no solutions). //