Lecture 11: More stuff about matrices

Recall the three equivalent statements about an $n \times n$ (square) matrix A:

(S1) $\text{rref}(A) = \mathbb{I}_n$

(S2) $\exists\, B$ with $AB = \mathbb{I}_n$

(S3) $\exists\, C$ with $CA = \mathbb{I}_n$.

If these hold, A is invertible, with inverse $A^{-1} := B = C$. This inverse matrix can be computed by taking

$$\text{rref}\left[A \mid \mathbb{I}_n\right] = \left[\mathbb{I}_n \mid A^{-1}\right].$$

There is a long list of additional conditions equivalent to (S1)–(S3) in the book. Here are a few of them:

(S4) $A\vec{x} = \vec{0}$ has only the trivial solution

is equivalent to all columns having a leading 1. Since A is square, this is the same as $\text{rref}(A) = \mathbb{I}_n$.

(S5) $A\vec{x}=\vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$

is equivalent to all rows having a leading '1',

which likewise is the same as $\operatorname{rref}(A) = I_n$ (because

A is square).

The corresponding statements for the linear transformation

$T: \mathbb{R}^n \to \mathbb{R}^n$ given by A are

(S4) T is 1-1

(S5) T is onto.

Since 1-1 and onto are different concepts, it may seem strange that these are equivalent statements here. Again, that's because the domain and codomain are both $\mathbb{R}^n$ (corresponding to the fact that A is $\underline{n \times n}$).

<u>Definition</u>: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> if there exists a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ with

$$(*) \quad \begin{cases} S(T(\vec{x})) = \vec{x} \\ T(S(\vec{x})) = \vec{x} \end{cases} \quad \text{for every } \vec{x} \in \mathbb{R}^n.$$

We write $T^{-1} := S$ (it's clear from (*) that this is unique).

<u>Theorem</u>: T is invertible $\iff$ its matrix A is invertible, and then T^{-1} has matrix A^{-1}.

<u>Proof</u>: If T is invertible, it is 1-1 and onto (otherwise (*) can't hold). By our equivalent conditions, A is invertible. Since composition of L.T.s corresponds to matrix multiplication, if $S(\vec{x}) := A^{-1}\cdot\vec{x}$ then $S(T(\vec{x})) = A^{-1}\cdot A\cdot\vec{x} = \mathbb{I}_n\cdot\vec{x} = \vec{x}$ and vice-versa $\Rightarrow S = T^{-1}$. $\square$

Our last equivalent condition, then, is

(S6) T is invertible.

Ex 1/ Is $A = \begin{pmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ invertible?

YES. No calculation required! You can see clearly that it has 4 pivots (hence will have $\mathrm{rref}(A) = \mathbb{I}_4$).

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Ex 2/ Is $A = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ invertible?

NO. You can see at once that it will have a row of all zeroes in its RREF. //

Ex 3/ Is rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin (thru angle θ) invertible as a L.T.?

YES. $R_{-\theta}$ is the inverse. You don't need to calculate matrices here. //

$\boxed{\text{LU Factorization}}$

We can apply what we've learned about elementary matrices & row-reduction to more general matrices A:

Suppose A can be reduced to a row-echelon matrix U entirely by replace operations that add multiples of rows to those below them. The elementary matrices which accomplish these operations are all of the form

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ a & & 1 \end{pmatrix}$$

with the "a" <u>below</u> the diagonal. We have

$$E_N \cdots E_1 A = U$$

$$A = (E_N \cdots E_1)^{-1} U = \underbrace{(E_1^{-1} \cdots E_N^{-1})}_{=: L} U$$

= lower triangular matrix with 1s on the diagonal

Here A need not be square (U will have the same dimensions), but L will always be.

Ex 4/ $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = A$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \downarrow$

$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix}$

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \cdot \downarrow$

$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix}$

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \cdot \downarrow$

$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = U$

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

So

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix},$$

and $A = L \cdot U$. //

Why is this useful? Suppose that you want to solve $A\vec{x} = \vec{b}$ for lots of $\vec{b}$'s.

<u>Approach 1</u>: $\vec{x} = A^{-1}\vec{b}$ (works if A is invertible)

<u>Approach 2</u>: $L(U\vec{x}) = \vec{b} \rightsquigarrow \begin{cases} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{cases}$

This is more efficient & general.

Ex 5/ A as above, $\vec{b} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$. For $L\vec{y} = \vec{b}$,

use $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{array}\right]$, get $\begin{cases} y_1 = -7 \\ y_2 = 5 + y_1 = -2 \\ y_3 = 2 - 2y_1 + 5y_2 = 2 + 14 - 10 = 6 \end{cases}$

For $U\vec{x} = \vec{y}$, use

$$\left(\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array}\right) \rightsquigarrow \begin{cases} x_3 = -6 \\ -2x_2 = x_3 - 2 = -8 \Rightarrow x_2 = 4 \\ 3x_1 = 7x_2 + 2x_3 - 7 = 9 \\ \quad \Rightarrow x_1 = 3 \end{cases}$$

So $\vec{x} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$. //

You can ignore the application in this section.

$\boxed{\text{Partitioned matrices}}$ (matrix multiplication in "blocks")

Say we have a 4×4 matrix A and a 4×2 matrix B.

$$
\underbrace{\left(\begin{array}{cc|cc}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
\hline
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{array}\right)}_{A}
\cdot
\underbrace{\left(\begin{array}{cc}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
\hline
b_{31} & b_{32} \\
b_{41} & b_{42}
\end{array}\right)}_{B}
=
\underbrace{\left(\begin{array}{cc}
c_{11} & c_{12} \\
c_{21} & c_{22} \\
\hline
c_{31} & c_{32} \\
c_{41} & c_{42}
\end{array}\right)}_{C}
$$

We could do the traditional row-by-column multiplication, which gives

$$c_{ij} = \sum_{k=1}^{4} a_{ik} b_{kj} .$$

It can be more convenient for some purposes to write

$$
\left(\begin{array}{c|c}
A_{11} & A_{12} \\
\hline
A_{21} & A_{22}
\end{array}\right)
\cdot
\left(\begin{array}{c}
B_1 \\
\hline
B_2
\end{array}\right)
=
\left(\begin{array}{c}
C_1 \\
\hline
C_2
\end{array}\right)
$$

so e.g. $C_1 = A_{11} \cdot B_1 + A_{12} \cdot B_2$,

matrix products

and c_{11} is obtained by the row-column multiplications shown:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}\begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}.$$

Your book has more examples & pictures of this.

Skip the business on "column-row" product expansions.

Ex 6/ Find formulas for X, Y, Z in terms of A, B, C, if

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline X & Y \end{array}\right] = \left[\begin{array}{c|c} 0 & I \\ \hline Z & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} A+BX & BY \\ \hline C & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} A+BX = 0 \quad \Rightarrow \quad BX = -A \\ C = Z \\ BY = I \end{cases}$$

$$\Rightarrow \begin{cases} Y = B^{-1} \\ Z = C \\ X = -B^{-1}A. \end{cases}$$

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