Lecture 12: Determinants

Suppose you were looking for a function

$$\det : \left\{ \begin{matrix} n \times n \text{ real} \\ \text{matrices} \end{matrix} \right\} \longrightarrow \mathbb{R}$$

with the following 3 properties, where we think of an $n \times n$ matrix A as an ordered n-tuple of row vectors $\begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{pmatrix}$:

(i) <u>Multilinearity</u>: $\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ a\vec{r}_i + b\vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = a \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} + b \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}$

(ii) <u>Antisymmetry</u>: $\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_n \end{pmatrix} = -\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}$

(iii) <u>Normalization</u>: $\det \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix} = 1$.

As we'll see, not only does such a function exist, it is unique. Anticipating this uniqueness, we'll call this function <u>the determinant</u>, written $\det(A)$ or $|A|$.

<u>Theorem 1</u>: If 2 rows of A are equal, $\det A = 0$.

<u>Proof</u>: If $\vec{r}_i = \vec{r}_j$, then by (ii), $\det A = -\det A$

$$\Rightarrow 2 \det A = 0 \Rightarrow \det A = 0. \qquad \square$$

$\boxed{2\times 2 \text{ matrices}}$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the defining properties (i) - (iii) to compute $\det A$:

<u>Case 1</u>: $a = c = 0$.
$$|A| = \begin{vmatrix} b\vec{e}_2 \\ d\vec{e}_2 \end{vmatrix} \underset{(i)}{=} bd \begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix} \underset{\text{Thm. 1}}{=} 0.$$

<u>Case 2</u>: $a = 0$.
$$|A| = \begin{vmatrix} b e_2 \\ c e_1 + d e_2 \end{vmatrix} \underset{(i)}{=} bc \begin{vmatrix} \vec{e}_2 \\ \vec{e}_1 \end{vmatrix} + bd \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix}}$$
$$= -bc \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \end{vmatrix} \underset{(iii)}{=} -bc.$$

<u>Case 3</u>: $a \neq 0$.
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \frac{c}{a} \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0} \underset{(i)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ c & \frac{bc}{a} \end{vmatrix}$$
$$\underset{(i)}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} \underset{(i)}{=} \begin{vmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d - \frac{bc}{a} \end{vmatrix}}_{0 \text{ by Case 1}} \underset{(i)}{=} a\left(d - \frac{bc}{a}\right) \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1 \text{ by (iii)}}$$
$$= ad - bc.$$

In each case, we get <u>$ad - bc$</u>.

<u>Theorem 2</u>: If A is an $n \times n$ upper (or lower) triangular matrix, then $\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$.

<u>Proof</u>: (will do for 3×3 matrices)
$$|A| = \begin{vmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{vmatrix} = \begin{vmatrix} \alpha\vec{e}_1 + a\vec{e}_2 + b\vec{e}_3 \\ \beta\vec{e}_2 + c\vec{e}_3 \\ \gamma\vec{e}_3 \end{vmatrix} = \alpha\beta\gamma \cancelto{1}{\begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}} + \alpha c\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_1 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}} + a\beta\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}}$$
$$+ ac\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}} + b\beta\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_3 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}} + bc\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_3 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}}$$
$$= \alpha\beta\gamma.$$
$\square$

$\boxed{3\times 3 \text{ matrices}}$ Let $A = \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \\ d & e & f \end{pmatrix}$. We have by (i)

$$\boxed{|A| = \alpha \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{vmatrix}}. \quad (*)$$

Now notice that

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} - a\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ d & e & f \end{vmatrix}}_{=0} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ d & e & f \end{vmatrix} - d\begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix}$$

by using Thm. 1 and (i) repeatedly.

In this way, $(*)$ becomes

$$\boxed{\alpha \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & 0 & c \\ d & 0 & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & 0 \\ d & e & 0 \end{vmatrix}} \quad (**)$$

Applying the same steps as for $n=2$ to the bottom 2 rows in $(**)$ gives:

$$\alpha(bf-ec)\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{|I_3|=1} + \beta(af-dc)\underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{\underset{(ii)}{=} -|I_3| = -1} + \gamma(ae-bd)\underbrace{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -(-|I_3|) = +1}$$

$$= \alpha \underbrace{\begin{vmatrix} b & c \\ e & f \end{vmatrix}}_{=: A_{\hat{1}\hat{1}}} - \beta \underbrace{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}_{=: A_{\hat{1}\hat{2}}} + \gamma \underbrace{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}_{=: A_{\hat{1}\hat{3}}},$$

which is the Laplace expansion of $\det(A)$ along the first row.

$\boxed{n\times n \text{ matrices}}$ Let $A_{\hat{i}\hat{j}} :=$ $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row & j^{th} column.

Theorem 3: For any fixed i,

"Laplace expansion in the i^{th} row"

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot \underbrace{(-1)^{i+j} \cdot \det(A_{\hat{i}\hat{j}})}_{=: C_{ij} = (i,j)^{\text{th}} \text{ cofactor}}.$$

Ex/ $\begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 3 \\ 0 & 2 & 0 \end{vmatrix} = (-1)^{2+1} 2 \begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = -2 \cdot 9 = -18.$ //

Another approach to 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{3} a_{1i} \vec{e}_i \\ \sum_{j=1}^{3} a_{2j} \vec{e}_j \\ \sum_{k=1}^{3} a_{3k} \vec{e}_k \end{vmatrix} = \sum_{i,j,k=1}^{3} a_{1i} a_{2j} a_{3k} \begin{vmatrix} \vec{e}_i \\ \vec{e}_j \\ \vec{e}_k \end{vmatrix}$$

0 unless i, j, k all different!

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

↑

(iii): the determinants of the permutation matrices shown are $(-1)^{\#\text{ of row swaps needed to get } I_3}$

$$= \text{“} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \text{”}$$

(visual mnemonic)

notice that this is also the # of column swaps needed to reach I_3 (you perform the same swaps in the reverse order).

⇓

The same definition of determinant but w/ column vectors replacing row vectors, yields the same answer! So we get

Theorem 3: $\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(A_{\hat{i}\hat{j}})$, for any fixed j;

(Laplace expansion in the j^{th} column)

and

Theorem 4: $\det A = \det A^T$.

Ex/
$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3\left(\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}\right) - 1\left(\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

$$= 3\cdot 4 - 1\cdot 0 = 12.$$

More on this next lecture.

An economic application: Leontief model

Here is the general problem: suppose the consumer sector of our economy demands, over the course of a year,

final demand vector
$$\vec{d} = \begin{pmatrix} 60 \\ 40 \\ 10 \end{pmatrix} \begin{array}{l} \text{units of manufacturing (MF)} \\ \text{units of transport (TR)} \\ \text{units of agriculture (AG)} \end{array}$$

Those sectors must actually produce <u>more</u> than this, since the aggies need their Deeres and the manufactured goods need to be transported to market. So to produce <u>one unit of output each</u>, the industries require the following <u>inputs</u> from each other:

require from

Consumption matrix

$$C = \begin{array}{ccc|c} MF & TR & AG & \\ \hline \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.5 & 0 & 0.2 \\ 0 & 0 & 0.2 \end{pmatrix} & & & \begin{matrix} MF \\ TR \\ AG \end{matrix} \end{array}$$

and so for a given output vector $\vec{x}$, we have the <u>intermediate demand</u>

$$C\vec{x}.$$

All told,

production vector

$$\vec{x} = C\vec{x} + \vec{d}$$

is the vector which tells how many units each industry needs to produce. As a linear system, this reads

$$(\mathbb{I}_3 - C)\,\vec{x} = \vec{d}.$$

To solve it, write either $\vec{x} = (I-C)^{-1}\vec{d}$, or row-reduce an augmented matrix. Row-reduction gives

$$\left[\begin{array}{ccc|c} 0.6 & -0.3 & -0.3 & 60 \\ -0.5 & 1 & -0.2 & 10 \\ 0 & 0 & 0.8 & 40 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -0.5 & -0.5 & 100 \\ -1 & 2 & -0.4 & 20 \\ 0 & 0 & 0.8 & 40 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -0.5 & -0.5 & 100 \\ 0 & 1.5 & -0.9 & 120 \\ 0 & 0 & 0.8 & 40 \end{array}\right]$$

$$\Rightarrow x_3 = \frac{40}{0.8} = \boxed{50} \Rightarrow x_2 = \frac{2}{3}(0.9\cdot 50 + 120) = \frac{2}{3}(165) = \boxed{110} \Rightarrow x_1 = \frac{110+50}{2} + 100 = \boxed{180}$$

— quite a bit bigger than $\vec{d}$. As for the inverse matrix, we write

$$(\mathbb{I}-C)(\mathbb{I}+C+C^2+C^3+\cdots+C^m)$$
$$= \mathbb{I} + \cancel{C + C^2 + \cdots + C^m}$$
$$\quad - \cancel{C - C^2 - \cdots - C^m} - C^{m+1}$$
$$= \mathbb{I} - C^{m+1} \approx \mathbb{I}$$ (C^{m+1} eventually very small if sums along columns of C are less than 1)

$$\Rightarrow (\mathbb{I}-C)^{-1} \approx \mathbb{I}+C+C^2+\cdots+C^m$$
$$\Rightarrow \vec{x} \approx \vec{d} + C\vec{d} + C^2\vec{d} + \cdots + C^m\vec{d} \quad \text{(for } m \text{ large)}.$$

The book provides a conceptual explanation for this form of the solution.