Lecture 15: Vector Spaces

The position $f(t)$ of a metal ball on a spring with strength k, being driven by an oscillating magnetic field of amplitude F, and encountering friction measured by r, can be described by the (differential) equation

(1) $$f''(t) + rf'(t) + kf(t) = F\cos(t).$$

Taking $r=0$ and $k=1$ for simplicity, we can rewrite (1) in terms of a <u>linear</u> differential operator T:

(2) $$\underbrace{\left(\frac{d^2}{dt^2} + 1\right)}_{=:T} f(t) = F\cos(t).$$

(why is it linear?)

I claim that — at least on the "space" of functions of the form

(3) $$f(t) = x_1\cos(t) + x_2\sin(t) + x_3 t\cos(t) + x_4 t\sin(t)$$

— we can think of T as a linear transformation

$$T: \mathbb{R}^4 \to \mathbb{R}^4.$$

The idea is to identify a function (3) with the vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, so that we get the table

vector $\vec{x} \in \mathbb{R}^4$ $\longleftrightarrow$	function $f(t)$ $\xrightarrow{T}$	function $T f(t)$ $\longleftrightarrow$	vector $T\vec{x}$
$\vec{e}_1$	$\cos t$	0	$\vec{0}$
$\vec{e}_2$	$\sin t$	0	$\vec{0}$
$\vec{e}_3$	$t \cos t$	$-2 \sin t$	$-2\vec{e}_2$
$\vec{e}_4$	$t \sin t$	$2 \cos t$	$2\vec{e}_1$

$$T : \mathbb{R}^4 \to \mathbb{R}^4$$

and T has standard matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The inhomogeneous ordinary differential equation, (2) becomes
(ODE)

$$A\vec{x} = \begin{pmatrix} F \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \longleftrightarrow \quad \left[\begin{array}{cccc|c} & O & & 2 & F \\ & & -2 & & 0 \\ & 0 & & O & 0 \\ 0 & & & & 0 \end{array}\right]$$

with solution $x_4 = \frac{F}{2}$, $x_3 = 0$, x_1 & x_2 free. So the general solution to (2) is

$$f(t) = \underbrace{c_1 \cos(t) + c_2 \sin(t)}_{\substack{\text{solutions to homogeneous equation} \\ A\vec{x} = \vec{0} \ / \ f'' + f = 0}} + \frac{F}{2} t \sin(t) .$$

The point of this example is that linear combinations (here, of functions) and linear transformations (here, $\frac{d^2}{dt^2} + 1$) occur throughout mathematics, and this makes it worth abstracting the properties of n-vectors in $\mathbb{R}^n$ and linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ to obtain more flexible concepts. (In particular, this will allow us to avoid constantly making identifications with $\mathbb{R}^n$.)

A (real) vector space, then, is a set V with a special element $\vec{0} \in V$, and three operations:

- ADDITION : $+ : V \times V \longrightarrow V$, $(\vec{v}, \vec{w}) \longmapsto \vec{v} + \vec{w}$
- SCALAR MULTIPLICATION : $\cdot : \mathbb{R} \times V \longrightarrow V$, $(r, \vec{v}) \longmapsto r\vec{v}$
- (ADDITIVE) INVERSION : $- : V \longrightarrow V$, $\vec{v} \longmapsto -\vec{v}$

which satisfy the following properties:

(i) "+" is commutative and associative

(ii) $\vec{0} + \vec{v} = \vec{v}$

(iii) $\vec{v} + (-\vec{v}) = \vec{0}$

(iv) $1\vec{v} = \vec{v}$

(v) $r(s\vec{v}) = (rs)\vec{v}$

(vi) distributivity: $\begin{cases} r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v} \\ (r+s)\vec{v} = r\vec{v} + s\vec{v}. \end{cases}$

The elements of V are called vectors.

Remarks on this definition:

(a) $0\vec{v} = (0+0)\vec{v} \underset{(vi)}{=} 0\vec{v} + 0\vec{v} \underset{(iii)}{\Longrightarrow} \vec{0} = 0\vec{v}$

(b) $(-1)\vec{v} = -\vec{v}$: because $\vec{v} + (-1)\vec{v} \underset{(iv)}{=} 1\vec{v} + (-1)\vec{v} \underset{(vi)}{=} (1+(-1))\vec{v}$

$= 0\vec{v} \underset{(a)}{=} \vec{0}$

(now use (iii) & (i))

So we don't separately specify what is "$-\vec{v}$" when constructing vector spaces: just the set V, and the operations "+" and "$\cdot$".

Examples

(1) $\mathbb{R}^m$ = m-tuples $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, "+" defined componentwise/etc.

(2) Directed line-segments in m-space (heretofore informally identified with (1)), "+" defined by parallelogram law/etc.

(3) $M_{m\times n}$ = $m\times n$ matrices with real entries
"+" = matrix addition, "$\cdot$" = scalar mult.
(matrix multiplication plays no role)

(4) $\mathbb{P}$ = polynomials with real coefficients
"+" = addition of polynomials
etc.

(5) $\mathcal{F}(S, \mathbb{R})$ = functions from a set S to $\mathbb{R}$
"+" adds values of functions pointwise: $(f+g)(s) = f(s) + g(s)$
"$\cdot$" multiplies " " " " by scalar: $(rf)(s) = r\,f(s)$

e.g. $\mathcal{F}(\mathbb{N}, \mathbb{R})$ = sequences
$\mathcal{F}(\mathbb{R}, \mathbb{R})$ = functions of 1 variable

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Non-Examples

① $\mathbb{R}^2$ with "+" defined by $(a_1,a_2)+(b_1,b_2) := (a_1+b_1, a_2-b_2)$
"$\cdot$" " " $r(a_1,a_2) := (ra_1, ra_2)$.

What is the problem? $-(a,b)$ exists and $= (a,b)$

But perhaps we should check distributivity:

$$(r+(-r))(a,b) = 0(a,b) = (0a, 0b) = (0,0)$$

while

$$r(a,b)+(-r)(a,b) = (ra, rb)+(-ra,-rb) = (ra+(-ra), rb-(-rb))$$
$$= (0, 2rb) \neq (0,0) \quad !$$

In other words, (vi) fails if $b \neq 0$, $r \neq 0$.
So this isn't a vector space.

② $\mathbb{R}^2$ with $\begin{cases} (a_1,a_2)+(b_1,b_2) := (a_1+b_1, 0) \\ r(a_1,a_2) := (ra_1, 0) \end{cases}$

Here the problem is (iv):

if $b \neq 0$, $(a,b) \neq (a,0) = 1(a,b)$.

Vector Subspaces

A subset $W \subseteq V$ of a vector space V is called a subspace if it is itself a (real) vector space, under the "+" and "·" inherited from V. Rather than check all the properties again, we can invoke the

Proposition: A nonempty subset $W \subseteq V$ is a (vector) subspace if and only if it contains all linear combinations $a\vec{w}_1 + b\vec{w}_2$ of its own elements $\vec{w}_1, \vec{w}_2 \in W$ — that is, iff W is closed under addition & scalar multiplication.

The "proof" is that (i) – (vi) are essentially inherited from V (and W contains $\vec{0}$ b/c you can take the trivial linear combination $0\vec{w}_1 + 0\vec{w}_2$). [To me, the Proposition is a mini-Theorem; to the book, it's the definition.]

$\boxed{\text{How to } \underline{\text{construct}} \text{ subspaces ?}}$

(A) Suppose we already have two: W_1 & W_2, in V. Then

- $W_1 \cap W_2$ (the intersection) is one
- $W_1 \cup W_2$ (the union) is <u>not</u> — e.g. consider

  $W_1 = x$-axis and $W_2 = y$-axis in $\mathbb{R}^2$;

  the union doesn't contain $(1,0) + (0,1)$.

(B) To fix this, define

$$W_1 + W_2 := \{ \vec{w}_1 + \vec{w}_2 \mid \vec{w}_1 \in W_1,\ \vec{w}_2 \in W_2 \} \subseteq V.$$

(C) In the same spirit, given vectors $\vec{v}_1, \dots, \vec{v}_n \in V$, define

$$W := \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \{ a_1\vec{v}_1 + \dots + a_n\vec{v}_n \mid a_1, \dots, a_n \in \mathbb{R} \} \subseteq V.$$

If $\operatorname{span}\{\vec{v}_i\}_{i=1}^{n} = V$ then we say "<u>the $\{\vec{v}_i\}$ span V</u>".

(D) Solutions, in $\mathbb{R}^n$, of $A\vec{x} = \vec{0}$ yield a subspace:

$$W = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n \ (= V \text{ here}).$$

[The same does not go for $A\vec{x} = \vec{b}$ (for some fixed $\vec{b} \neq \vec{0}$): If $\vec{x}_1, \vec{x}_2$ both solve $A\vec{x} = \vec{b}$, then does $\vec{x}_1 + \vec{x}_2$? No, it solves $A\vec{x} = 2\vec{b}$. So the solution set isn't closed under "+".]

(E) $C^0(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$ (continuous functions)

$\mathbb{P}_n \subseteq \mathbb{P}$ (polynomials of degree $\leq n$)

$\left\{ \begin{matrix} \text{upper triangular} \\ \text{matrices} \end{matrix} \right\} \subseteq M_{n \times n}$.

The book has other interesting examples and you should have a look.