Lecture 16: Kernel & Image

The last lecture illustrated how we can abstractify the properties of $\mathbb{R}^m$ to obtain the notion of a vector space. This allows us to do away with the formality of identifying with $\mathbb{R}^m$ when performing computations. The next step is to bring our notion of transformations up to this level.

Definition 1: A linear transformation is a map (function) between two (not necessarily different) vector spaces

$$T : V \to W$$

obeying linearity: $T(a\vec{v}_1 + b\vec{v}_2) = a\,T(\vec{v}_1) + b\,T(\vec{v}_2)$.
(Note that $T(\vec{0}) = \vec{0}$ is an immediate consequence of this definition.)

Examples

(1) $V = \mathbb{R}^n$, $W = \mathbb{R}^m$

$T(\vec{v}) := A\vec{v}$ (we know this is linear already)

(2) $V = W = \mathbb{P}_3$ (polynomials of degree ≤ 3)

$T = \frac{d}{dt}$ is linear, since $\frac{d}{dt}(aP + bQ) = a\frac{dP}{dt} + b\frac{dQ}{dt}$

(3) $V = W = C^0(\mathbb{R})$ (real-valued continuous functions on $\mathbb{R}$)

T sends $f(x) \longmapsto (Tf)(x) := \int_0^x f(t)\,dt$

(4) $V = \mathbb{R}^4$, $W = \mathcal{F} :=$ space of functions spanned by $\cos(t), \sin(t), t\cos(t), t\sin(t)$

In Lecture 15, we had $T: \mathcal{F} \to \mathcal{F}$ given by $\frac{d^2}{dt^2} + 1$.

To make identifications with $\mathbb{R}^4$, we used a different linear transformation, $S: \mathbb{R}^4 \to \mathcal{F}$ sending

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto x_1 \cos t + x_2 \sin t + x_3 t\cos t + x_4 t\sin t =: f_{\vec{x}}(t)$$

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Two kinds of subspaces

Let $T: V \to W$ be a linear transformation.

Definition 2: The image (or range) of T is the set of all vectors in W that are "hit" by T:

$$\operatorname{im}(T) := \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\} \subseteq W$$

Definition 3: The kernel of T is the set of vectors in V "killed" by T:

$$\ker(T) := \{\vec{v} \in V \mid T\vec{v} = \vec{0}\} \subseteq V.$$

Examples

(1) $T : \mathbb{R}^n \to \mathbb{R}^m$ has
$\vec{x} \longmapsto A\vec{x}$

$$\ker(T) = \{x \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} =: \underline{\mathrm{Nul}}(A) \subseteq \mathbb{R}^n$$ null space of A

$$\mathrm{im}(T) = \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\} =: \underline{\mathrm{Col}}(A) \subseteq \mathbb{R}^m$$ column space of A = span of A's columns

(2) $\frac{d}{dt} : \mathbb{P}_3 \to \mathbb{P}_3$

$$\ker\left(\tfrac{d}{dt}\right) = \left\{P \in \mathbb{P}_3 \;\middle|\; \tfrac{dP}{dt} = 0\right\} = \mathbb{P}_0 \;(= \mathbb{R}) \text{ is constant functions}$$

$$\mathrm{im}\left(\tfrac{d}{dt}\right) = \left\{Q \in \mathbb{P}_3 \;\middle|\; Q = \tfrac{dP}{dt} \text{ for some } P \in \mathbb{P}_3\right\} = \mathbb{P}_2$$

(3) "$\int_0^x$" $= T : C^0(\mathbb{R}) \to C^0(\mathbb{R})$

$$\ker(T) = \left\{f \in C^0(\mathbb{R}) \;\middle|\; \int_0^x f(t)\,dt \text{ is } \underset{\text{for all } x}{\text{identically}}\ 0\right\} = \{0\}$$

$$\mathrm{im}(T) = \left\{g \in C^0(\mathbb{R}) \;\middle|\; g(x) = \int_0^x f(t)\,dt \text{ for some } f \in C^0(\mathbb{R})\right\} = \underline{C^1(\mathbb{R})_0}$$

(by the FTC)

continuously differentiable functions with $g(0)=0$

(4) $S : \mathbb{R}^4 \to \mathcal{F}$

$$\ker(S) = \{x \in \mathbb{R}^4 \mid f_{\vec{x}}(t) = 0 \text{ (for all } t)\} = \{\vec{0}\}$$

$$\mathrm{im}(S) = \{f \in \mathcal{F} \mid f = f_{\vec{x}} \text{ for some } \vec{x}\} = \mathcal{F}$$

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Let $T: V \to W$ be a linear transformation.

Proposition A: $\mathrm{im}(T) \subseteq W$ and $\ker(T) \subseteq V$ are subspaces.

Proof: If $w, w' \in \mathrm{Im}(T)$, then $\begin{cases} w = T(v) \\ w' = T(v') \end{cases}$ for some $v, v' \in V$.

By linearity, $T(av + bv') = aTv + bTv' = aw + bw'$

$\Rightarrow aw + bw' \in \mathrm{im}(T)$.

If $v, v' \in \ker(T)$, then $\begin{cases} Tv = 0 \\ Tv' = 0 \end{cases} \Longrightarrow$

$T(av + bv') = aTv + bTv' = a \cdot 0 + b \cdot 0 = 0 \Rightarrow$

$av + bv' \in \ker(T)$. $\square$

Definition 4: (i) T is onto if $\mathrm{im}(T) = W$

(ii) T is 1-1 if $v \neq v' \Rightarrow Tv \neq Tv'$

(i.e. $Tv = Tv' \Rightarrow v = v'$)

Proposition B: T is 1-1 $\Leftrightarrow \ker(T) = \{0\}$.

Proof: ($\Longrightarrow$) is clear: only 0 can go to 0.

($\Longleftarrow$) Suppose $\ker(T) = \{0\}$, and let $Tv = Tv'$.

Then by linearity, $0 = Tv - Tv' = T(v - v')$ so

$v - v' \in \ker(T)$ $(= \{0\})$ $\Rightarrow v - v' = 0 \Rightarrow v = v'$. $\square$

Examples /

1. T is 1-1 $\iff$ $Nul(A) = \{0\}$; T is onto $\iff$ $Col(A) = \mathbb{R}^m$
2. $\frac{d}{dt}$ is neither 1-1 nor onto
3. T is 1-1 but not onto
4. S is 1-1 and onto //

Two kinds of problems

How do we find Null and Column spaces of a matrix A?

I. Let $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$. (a) Is $\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \\ 1 \\ 0 \end{pmatrix}$ in Nul(A)? (b) Parametrically describe Nul(A).

(a) Yes : just compute $A\vec{v} = \vec{0}$

(b) We want all the solutions of $A\vec{x} = \vec{0}$, and we know how to describe them parametrically : x_2, x_4 are non-pivot variables and thus free ; we have

$$\begin{cases} x_1 = 4x_2 - 2x_4 \\ x_3 = 5x_4 \\ x_5 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4x_2 - 2x_4 \\ x_2 \\ 5x_4 \\ x_4 \\ 0 \end{pmatrix}$$

$$\Rightarrow Nul(A) = \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} \middle| x_2, x_4 \in \mathbb{R} \right\} \subseteq \mathbb{R}^5.$$

(i.e. the span)

So it's easy to check if a given vector is in Nul(A), but requires work to construct vectors in Nul(A).

II Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 1 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$. (a) Is $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ in Col(A)?

(b) Parametrically describe Col(A).

(a) Need to see if $A\vec{x} = \vec{v}$ is consistent:

$$\left[\begin{array}{cc|c} 6 & -4 & 0 \\ -3 & 1 & 1 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{3 scale}} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 1 & 1 \\ -3 & 2 & 0 \\ -3 & 2 & 0 \end{array}\right] \xrightarrow[\text{ops.}]{\text{3 replace}} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which yields $\begin{cases} -x_2 = 1 \\ 3x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1 \end{pmatrix}$.

So, Yes.

(b) easy: $\mathrm{Col}(A) = \left\{ s\begin{pmatrix} 6 \\ -3 \\ -9 \\ 9 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \\ 6 \\ -6 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$

In contrast to Nul(A), it's easy to construct vectors in Col(A), but requires work to check if a given vector is in Col(A).