Lecture 18: Coordinates w.r.t. a basis

Coordinate Vectors Let V be a vector space, and $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \subset V$ a basis. Take any $\vec{x} \in V$.

<u>Theorem 1</u>: We can write $\vec{x}$ as a sum $\sum_i \alpha_i \vec{b}_i$ in exactly one way.

<u>Proof</u>: Since $\mathcal{B}$ spans V, we can write $\vec{x}$ as such a sum. If $\sum_{i=1}^{n} \alpha_i \vec{b}_i = \vec{x} = \sum_{i=1}^{n} \gamma_i \vec{b}_i$, then $\vec{0} = \sum_{i=1}^{n} (\alpha_i - \gamma_i)\vec{b}_i$. Since $\mathcal{B}$ is a linearly independent set, $\alpha_i - \gamma_i = 0$ for all i; that is, $\alpha_i = \gamma_i$. $\square$

We write $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} =:$ the <u>coordinate vector</u> of $\vec{x}$ with respect to $\mathcal{B}$.

Notice that the procedure of "taking the coordinate vector" gives a map

$$[\cdot]_{\mathcal{B}} : V \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}.$$

<u>Theorem 2</u>: $[\cdot]_{\mathcal{B}}$ is a linear transformation which is also 1-1 and onto, i.e. an <u>isomorphism</u>.

Proof: If $\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{b}_i$ & $\vec{y} = \sum_{i=1}^{n} \beta_i \vec{b}_i$, then $c\vec{x} = \sum c\alpha_i \vec{b}_i$
& $\vec{x}+\vec{y} = \sum (\alpha_i + \beta_i) \vec{b}_i$.

So $[c\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c\alpha_1 \\ \vdots \\ c\alpha_n \end{pmatrix} = c \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = c[\vec{x}]_{\mathcal{B}}$

and $[\vec{x}+\vec{y}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1+\beta_1 \\ \vdots \\ \alpha_n+\beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$

$\Longrightarrow [\cdot]_{\mathcal{B}}$ linear.

If $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, then $\vec{x} = 0\vec{b}_1 + \cdots + 0\vec{b}_n = \vec{0}$ ($\Longrightarrow [\cdot]_{\mathcal{B}}$ is 1-1)

Any $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is of the form $[\alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n]_{\mathcal{B}}$ ($\Longrightarrow [\cdot]_{\mathcal{B}}$ is onto). $\square$

Linear transformations which are isomorphisms ($T: V \xrightarrow{\cong} W$ is the notation) have inverses which are also linear transformations. They therefore produce an exact correspondence between linear combinations, linear independence/dependence, etc. in the two vector spaces.

Ex 1/ $[\cdot]_{\mathcal{B}} : \mathbb{P}_n \xrightarrow{\cong} \mathbb{R}^{n+1}$, $\mathcal{B} = \{1, t, \ldots, t^n\}$.

$$a_0 + a_1 t + \cdots + a_n t^n \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

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Ex 2/ $[\cdot]_{\mathcal{B}} : W \xrightarrow{\cong} \mathbb{R}^4$, where $W = \text{span}\{\underbrace{\cos t, \sin t, t\cos t, t\sin t}_{\mathcal{B}}\}$

$$a\cos t + b\sin t + c\,t\cos t + d\,t\sin t \longmapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

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Ex 3 / $A = m \times n$ matrix with k non-pivot columns.

$V = \mathrm{Nul}(A) \subset \mathbb{R}^n$

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ the basis $\underline{\text{of } V}$ produced by our procedure in Lecture 17

$$[\cdot]_{\mathcal{B}} : V \xrightarrow{\cong} \mathbb{R}^k$$

More concretely, the plane in $\mathbb{R}^{3\,(=n)}$ described by $4x_1 - 5x_2 + 2x_3 = 0$ is a subspace of this type, with

$$A = \begin{bmatrix} 4 & -5 & 2 \end{bmatrix}.$$

We have $\mathcal{B} = \left\{ \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5/4 \\ 1 \\ 0 \end{pmatrix} \right\}$, and $[\cdot]_{\mathcal{B}}$ gives an isomorphism to $\mathbb{R}^2$. //

$\boxed{\text{Bases of } \mathbb{R}^n}$ Any basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_k\} \subset \mathbb{R}^n$ has an associated $n \times k$ matrix

$$P_{\mathcal{B}} := \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_k \\ \downarrow & & \downarrow \end{pmatrix}.$$

Since $\mathcal{B}$ is linearly independent, $\mathrm{rref}(P_{\mathcal{B}})$ has a leading 1 in each column $\implies k \leq n$.

Since $\mathcal{B}$ spans $\mathbb{R}^n$, rref$(P_\mathcal{B})$ has a leading 1 in each row $\implies k \geq n$.

So $\underline{k = n}$, and since the columns of $P_\mathcal{B}$ span $\mathbb{R}^n$,

$$\underline{P_\mathcal{B} \text{ is invertible.}}$$

What does this matrix have to do with anything, though?

Ex 4/ $\mathcal{B} = \left\{ \overset{\vec{b}_1}{\begin{pmatrix} 5 \\ -5 \end{pmatrix}}, \overset{\vec{b}_2}{\begin{pmatrix} 5 \\ -1 \end{pmatrix}} \right\}$, $[\vec{x}]_\mathcal{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$. What is $\vec{x}$?

$$\vec{x} = 6\vec{b}_1 + 4\vec{b}_2 = 6\begin{pmatrix} 5 \\ -5 \end{pmatrix} + 4\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 50 \\ -36 \end{pmatrix}, \text{ done.}$$

But notice: $\vec{x} = \begin{pmatrix} 5 & 5 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = P_\mathcal{B}[\vec{x}]_\mathcal{B}$. //

Indeed, if $[\vec{x}]_\mathcal{B} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, then by definition

$$\vec{x} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = P_\mathcal{B}[\vec{x}]_\mathcal{B}.$$

In fact, "$\vec{x}$" is <u>itself</u> a coordinate vector — with respect to the standard basis $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$:

$$\vec{x} = [\vec{x}]_\mathcal{E}.$$

So we call $P_\mathcal{B}$ the <u>change-of-basis</u> (or change-of-coordinates) <u>matrix from $\mathcal{B}$ to $\mathcal{E}$</u>.

Of course, you don't need a matrix to go from $[\vec{x}]_{\mathcal{B}}$ to $\vec{x}$, as we see in the Example above. But it's great for going in the other direction.

Ex 5/ $\mathcal{B} = \left\{ \overset{\vec{b}_1}{\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}}, \overset{\vec{b}_2}{\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}}, \overset{\vec{b}_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$, $\vec{x} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

So: $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x}$, and we have 2 options: row-reduce $[P_{\mathcal{B}} \mid \vec{x}]$, or use $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x}$. I'll go the second route:

$$\left[\underbrace{\begin{matrix} 1 & & \\ 4 & 1 & \\ 4 & -4 & 1 \end{matrix}}_{P_{\mathcal{B}}} \middle| \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}\right] \longrightarrow \cdots \longrightarrow \left[\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 1 & & \\ -4 & 1 & \\ -20 & 4 & 1 \end{matrix}}_{P_{\mathcal{B}}^{-1}}\right]$$

$$\Rightarrow [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ -20 & 4 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ -63 \end{pmatrix}. \quad //$$

Ex 6/ Write $p(t) = 1 + 5t - 2t^2$ with respect to the basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$ of $\mathbb{P}_2$.

Identify $\mathbb{P}_2$ with $\mathbb{R}^3$ via $a_0 + a_1 t + a_2 t^2 \leftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

Then $\vec{x} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$, $P_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow P_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix}$

$$\Rightarrow [p(t)]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}.$$

[<u>Check</u>: $4 \cdot 1 + 1 \cdot (t-1) + (-2)\cdot(t-1)^2 = 4 + t - 1 - 2t^2 + 4t - 2 = 1 + 5t - 2t^2$.] //