Lecture 2: Row reduction

Using the "Replace", "Swap", and "Scale" row-operations, we shall now discuss how to put any matrix in a particularly nice form:

Reduced Row-Echelon Form (RREF)

A matrix A is in RREF if all of the following hold:

(i) the first nonzero entry of each row is 1, called a "leading 1";

(ii) when a column contains a leading 1, all other entries in that column are 0 (this is called a "pivot column"); and

(iii) when a row contains a leading 1, each row above it contains a leading 1 to the left.

The weaker notion of Row-Echelon Form (REF) is obtained by dropping (i) and weakening (ii) and (iii):

(ii′) when a column contains the first nonzero entry of some row, all the entries of the column below it are 0 ; and

(iii′) the leading nonzero entry of a row occurs further to the right than all the leading entries in the rows above it.

Ex 1 / If "$*$" stands for "arbitrary numbers", then

$$\begin{pmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ are in}$$

reduced row-echelon form (RREF),

while (if "$\bullet$" stands for "arbitrary NONZERO number")

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \end{pmatrix} \text{ are in}$$

row-echelon form (REF). //

Row reduction means "some procedure for associating an RREF matrix to a given matrix A":

$$A \longmapsto \mathrm{rref}(A).$$

Since the procedure simply applies row operations to A, the new matrix is row-equivalent to A.

$$\boxed{\text{FACT: There is exactly one RREF matrix row-equivalent to a given matrix } A.}$$

(We'll explain why in Lecture 3.)

CONSEQUENCE: rref(A) is independent of the procedure used!

The book has a 2-stage row-reduction algorithm:

- convert A to a REF matrix
- convert the REF matrix to a RREF one.

(See the "appendix" below.) Here's a simpler version:

Row-reduction algorithm

(a) "cursor" starts at upper left-hand entry of matrix;

(b) move cursor to right if the cursor entry and all entries below it are 0; repeat until this is no longer the case;

(c) if cursor entry = 0, swap cursor row with the first row below it having nonzero entry in the cursor column; SWAP

(d) divide cursor row by cursor entry; SCALE

(e) eliminate all other entries in the cursor column by adding suitable multiples of the cursor row to other rows; REPLACE

(f) if cursor is at bottom right, STOP.
Otherwise, move down & to the right, and go back to (b).

Let's illustrate with an example: $\boxed{\ }$ = cursor

Ex 2/

$$A = \begin{pmatrix} \boxed{0} & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} \boxed{2} & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{(e)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix} = \mathrm{rref}(A).$$

//

Now, how do we use this to solve a linear system?

STEP 1: Convert the system to an augmented matrix

$$M = [A \mid \vec{b}]$$

STEP 2: Apply the above algorithm to compute

$$\mathrm{rref}(M).$$

STEP 3: Convert back to a linear system and find the tuples $(x_1, \dots, x_n)$ solving it. (More precisely: use the non-pivot variables to parametrize the solution set. This is sometimes called "back substitution".)

Ex 3/ $\begin{cases} 3x_1 - 6x_2 + 2x_3 - x_4 = 1 \\ -2x_1 + 4x_2 + x_3 + 3x_4 = 4 \\ \phantom{-2x_1 + 4x_2 +{}} x_3 + x_4 = 2 \\ x_1 - 2x_2 + x_3 \phantom{{}+ x_4} = 1 \end{cases}$

STEP 1

$$M = \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 1 \end{array}\right] \xrightarrow[\rho_1 \mapsto \frac{1}{3}\rho_1]{\text{scale}} \left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\begin{cases}\rho_2 \mapsto \rho_2 + 2\rho_1 \\ \rho_4 \mapsto \rho_4 - \rho_1\end{cases}]{\text{replace}} \left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 7/3 & 7/3 & 14/3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right] \xrightarrow[\rho_2 \mapsto \frac{3}{7}\rho_2]{\text{scale}} \left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right]$$

STEP 2

$$\xrightarrow[\begin{cases}\rho_3 \mapsto \rho_3 - \rho_2 \\ \rho_4 \mapsto \rho_4 - \frac{1}{3}\rho_2 \\ \rho_1 \mapsto \rho_1 - \frac{2}{3}\rho_2\end{cases}]{\text{replace}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \operatorname{rref}(M)$$

STEP 3

$$\left.\begin{array}{r} x_1 - 2x_2 \qquad - x_4 = -1 \\ x_3 + x_4 = 2 \end{array}\right\}$$

Solve for the variables corresponding to pivot columns, called basic variables: $\begin{cases} x_1 = 2x_2 + x_4 - 1 \\ x_3 = 2 - x_4 \end{cases}$

The free variables are the non-pivot ones, and are so named because they can be freely chosen. They parametrize the solution set of the linear system:

$$\mathcal{S} = \{(2x_2 + x_4 - 1,\ x_2,\ 2 - x_4,\ x_4) \mid x_2, x_4 \in \mathbb{R}\}.$$

Ex 4 /

$$\begin{cases} 3x_1 - 6x_2 + 2x_3 - x_4 = 1 \\ -2x_1 + 4x_2 + x_3 + 3x_4 = 4 \\ \qquad\qquad\quad x_3 + x_4 = 2 \\ x_1 - 2x_2 + x_3 \qquad = 0 \end{cases}$$

same as Ex. 3 except for this

$$\left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\text{Ex. 3}]{\text{steps from}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

swap $\rho_3 \leftrightarrow \rho_4$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\rho_3 \mapsto -1\cdot\rho_3]{\text{scale}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

replace $\rho_1 \mapsto \rho_1 + \rho_3$, $\rho_2 \mapsto \rho_2 - 2\rho_3$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 - 2x_2 \quad - x_4 = 0 \\ \qquad\qquad x_3 + x_4 = 0 \\ \qquad\qquad \boxed{0 = 1} \end{cases}$$

system inconsistent!

Solution set $\mathcal{S} = \emptyset \Longleftarrow$

From the last example, we notice that

- If the last column of the augmented matrix is a pivot column, then the system is inconsistent.

Otherwise, steps 1-3 tell us how to solve the system, and so:

- If the last column is non-pivot, the system is consistent; it has a unique solution if there are no free variables, i.e. if all but the last column are pivots.

So far we have looked at m equations in n unknowns, with $m = n$. What about the other cases?

- If $m > n$, the system is called overdetermined.
- If $m < n$, the system is called underdetermined.

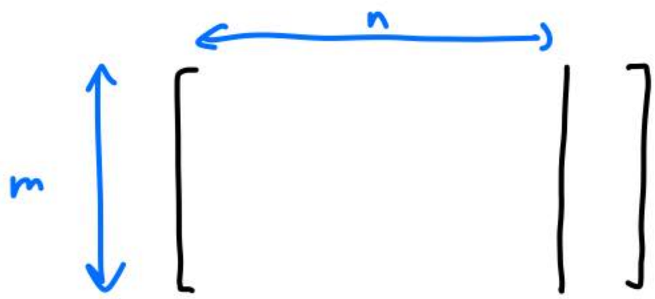
Q : Can an overdetermined system be consistent?

A : YES. But some equations will have to be "linear combinations" of others.

Q : Can an underdetermined system have a unique solution?

A : NO. Think about intersections of planes in space: two planes in 3-space will never intersect in a point.

A more mathematical answer can be formulated as follows: when you reduce an augmented matrix of the form



$(m < n)$

to RREF, there must be non-pivot columns, hence free variables. This is because each pivot column contains a leading 1 for some row, and there are only m rows hence $\leq n$ leading 1's. In general, the number of pivots is at most the smaller of m & n.

APPENDIX (The book's row-reduction algorithm)

PART I Produce REF matrix:

I(a): "cursor" starts at upper left-hand entry.

I(b): if necessary, move cursor to right until you reach a nonzero column.

I(c): **SWAP** if cursor entry = 0, exchange cursor row with 1st row below it having nonzero entry in cursor column.

I(d): **REPLACE** eliminate entries below the cursor (in cursor column) by adding multiples of cursor row to rows below.

I(e): move the cursor down & to the right, hide all rows above it & columns to the left of it, go back to I(b).

$\boxed{\text{PART II}}$ REF to RREF :

II (a) : "Cursor" starts at right most $\overbrace{\text{leading entry}}^{\text{first nonzero entry in a row}}$

II (b) : divide cursor row by cursor entry
SCALE

II (c) : eliminate entries above cursor (by the "replace" operation)
REPLACE

II (d) : move cursor left & up to the next leading entry,
go back to II (b).

Again, stop when you exit the matrix.