

Lecture 20: Rank + Nullity

Let A be an $m \times n$ matrix. In particular, A has n columns. Recall that the pivot columns of A furnish a basis of the column space $\text{Col}(A) \subseteq \mathbb{R}^m$; while the null space $\text{Nul}(A) \subseteq \mathbb{R}^n$ (comprising solutions of $A\vec{x} = \vec{0}$) has a basis whose elements correspond to the non-pivot variables (since they parametrize the general solution).

Definition: (a) The rank of A is

$$\begin{aligned}\text{rank}(A) &:= \dim(\text{Col}(A)) = \# \text{ of pivot columns} \\ &= \# \text{ of leading 1's in } \text{ref}(A).\end{aligned}$$

(b) The nullity of A is

$$\begin{aligned}\text{nullity}(A) &:= \dim(\text{Nul}(A)) = \# \text{ of non-pivot variables} \\ &= \# \text{ of non-pivot columns.}\end{aligned}$$

Theorem 1: $\text{rank}(A) + \text{nullity}(A) = n$.

Proof: ($\#$ of pivot columns) + ($\#$ of non-pivot columns)
 $= \#$ of columns! □

Ex 1 / Suppose that a homogeneous linear system $A\vec{x} = \vec{0}$ of 8 equations in 15 unknowns has (exactly) 10 independent solutions. If we want to turn this into a consistent inhomogeneous system $A\vec{x} = \vec{b}$, how many parameters govern the allowable choices for \vec{b} ? Of course, \vec{b} must lie in $\text{Col}(A)$, whose dimension = $\text{rank}(A) = 15 - \text{nullity}(A) = 15 - 10 = \underline{\underline{5}}$. //

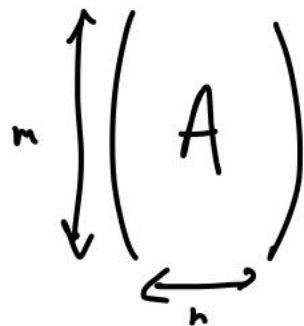
Q: What is the largest possible rank of an $m \times n$ matrix?

Well, $\text{rank}(A) = \#$ of pivot columns
 $= \#$ of leading 1's in $\text{rref}(A)$
 $= \#$ of nonzero rows in $\text{rref}(A)$

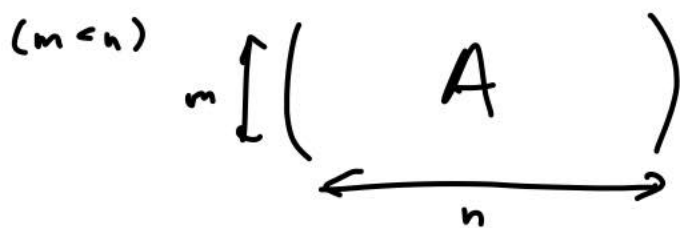
So $\text{rank}(A) \leq \min\{m, n\}$. When "=", we say that A has "maximal rank".

Q: What does maximal rank "look like"? 3 poss cases:

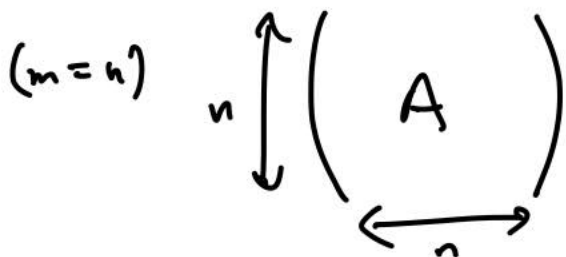
$(m > n)$



If $\text{rank}(A) = n$, then
 (by Thm. 1) $\text{nullity}(A) = 0$. So
 the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$
 $\vec{x} \mapsto A\vec{x}$
 is 1-to-1.



If $\text{rank}(A) = m$, then
 $m = \dim(\text{Col}(A))$
 $= \dim(\text{image}(T))$,
 and so T is onto.



If $\text{rank}(A) = n$, then
 $\text{nullity}(A) = 0 \implies A$ invertible
 $\implies T$ is 1-1 & onto (isomorphism).

Remark: (i) For square ($n \times n$) matrices, " $\text{rank}(A) = n$ " (equivalently, " $\text{nullity}(A) = 0$ ") is yet another condition equivalent to invertibility of A .

(ii) For non-square matrices, maximal rank is the closest you can get to invertibility. On p.1 of the recent exam (where $AB = I_m$ with A, B non-square), A & B both must be of maximal rank.

Row spaces

Let A be an $m \times n$ matrix.

Definition: $\text{Row}(A) \subset \mathbb{R}^n$ is the subspace consisting of all linear combinations of rows of A . (Can also think of this as $\text{Col}(A^T)$.) Row space is not changed by elementary row operations, so $\text{Row}(A) = \text{Row}(\text{ref}(A))$.

Ex 2 / $A = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \end{pmatrix}$. Find bases & dimensions of

$\text{Col}(A)$, $\text{Nul}(A)$, $\text{Row}(A) (= \text{Col}(A^T))$, $\text{Nul}(A^T)$.

\uparrow dependencies on columns of A \uparrow dependencies on rows of A .

[We could flip A into A^T and compute $\text{Col}(A^T)$ & $\text{Nul}(A^T)$ directly, by row-reducing A^T . But there is a way to do all four just by row-reducing A .]

$$A \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

\swarrow pivot column \searrow

So a basis for $\text{Col}(A) = 1^{\text{st}} + 2^{\text{nd}}$ columns of $A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$
 $\implies \text{rank}(A) = 2 = \dim(\text{Col}(A))$.

and a basis for $\text{Nul}(A) = \left(\begin{array}{l} \text{Solutions to } \begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \\ \text{with } \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \right) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\implies \text{nullity}(A) = 3 = \dim(\text{Nul}(A))$.

A basis for $\text{Row}(A) = \text{Row}(\text{rref}(A))$ is $\{(1 \ 0 \ 1 \ 1 \ 0), (0 \ 1 \ 1 \ 0 \ 0)\}$
 $\implies \dim(\text{Row}(A)) = 2$

while for $\text{Nul}(A^T)$, the basis is just $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ since $-\frac{1^{\text{st}}}{\text{row of } A} + \frac{3^{\text{rd}}}{\text{row of } A} = \vec{0}$
 $\implies \dim(\text{Nul}(A^T)) = 1$.

Notice that $2+3=5$ is "rank + nullity" for A , while
 $2+1=3$ is "rank + nullity" for A^T .

Remark: In general, you could find a basis for $\text{Nul}(A^T)$ by using augmented matrices:

$$[A \mid I_3] \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \boxed{-1 \ 2 \ 0} \\ \boxed{1 \ -1 \ 0} \\ \boxed{-1 \ 0 \ 1} \end{array}$$

these keep a record of row operations:

The rows in the 3×3 to the right of the bar following rows of all 0's in $\text{ref}(A)$, yield a basis for $\text{Nul}(A^T)$.

$$\begin{aligned} \text{row 1}(\text{ref}) &= -\text{row 1}(A) + 2 \text{row 2}(A) \\ \text{row 2}(\text{ref}) &= \text{row 1}(A) - \text{row 2}(A) \\ 0 &= \text{row 3}(\text{ref}) = -\text{row 1}(A) + \text{row 3}(A) \end{aligned}$$

dependency //
(this is why)

Theorem 2: $\dim(\text{Row}(A)) = \text{rank}(A) (= \dim(\text{Col}(A)))$

Proof: The nonzero rows of $\text{ref}(A)$ are a basis for $\text{Row}(A)$.

They clearly span it, by reversing the elementary row operations.

They are independent b/c the 'leading 1' in each nonzero row occurs in a different column, as the sole nonzero entry in that column.

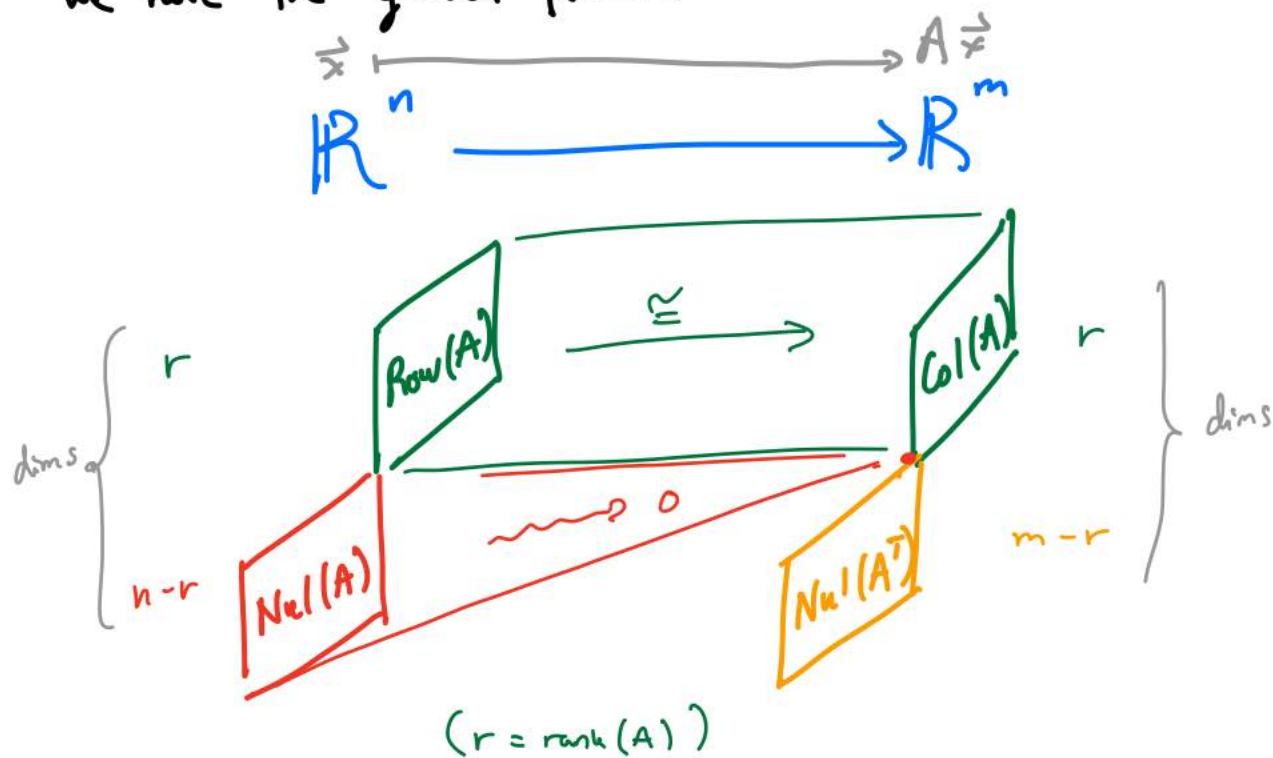
So $\dim(\text{Row}(A)) = \# \text{ nonzero rows} = \# \text{ leading 1s} = \text{rank}(A)$. \square

Another way of saying this is that A & A^T always have the same rank.

Ex 3 / Consider a homogeneous system of 12 linear equations in 8 unknowns with 2 independent solutions (& no more).
Can we reduce the # of equations?

Represent as $A\vec{x} = \vec{0}$, A 12×8 ; a basis for $\text{Row}(A)$ corresponds to a "minimal" set of equations. The number thereof = $\text{rank}(A) = 8 - \text{nullity}(A) = 8 - 2 = 6$. So indeed, we can eliminate half of the original equations. //

To summarize our discussion of the 4 subspaces above, we have the general picture



and one can in fact show that $\text{Row}(A)$ is mapped onto $\text{Col}(A)$ in 1-to-1 fashion.

Linear Transformation version

Let $T: V \rightarrow W$ be a linear transformation between two (finite-dimensional) vector spaces. Identifying V & W with \mathbb{R}^n & \mathbb{R}^m , so that T has an $m \times n$ matrix A , allows us to reinterpret Theorem 1:

Theorem 1': $\underbrace{\dim(\text{Im}(T))}_{\text{rank}(A)},$ $\underbrace{\dim(\text{Ker}(T))}_{\text{nullity}(A)} = \dim(V).$
 since $\text{Im}(T)$ identifies
 w/ $\text{Col}(A)$

Naturally, we call $\dim(\text{Im}(T))$ the rank of T and
 $\dim(\text{Ker}(T))$ the nullity of T .

Ex 4/ If $T: V \xrightarrow{\cong} W$ is an isomorphism, then
 $\text{Ker}(T) = \{0\}$ & $\text{Im}(T) = W$. But then Thm. 1' says
 $\dim W = \dim V$. //

Ex 5/ If $T = \frac{d}{dt}: \mathbb{P}_3 \rightarrow \mathbb{P}_3$, nullity = 1 & rank = 3.
 (Ker(T) = constants) (Im(T) = \mathbb{P}_2)
 Indeed $1+3 = 4 = \dim \mathbb{P}_3$. //

Ex 6/ If $T = \left(\frac{d}{dt}\right)^2 + 1: W \rightarrow W$, where W is the
 4-dim'l space of functions spanned by $\cos(t), \sin(t), t\cos(t), t\sin(t)$,
 then we found we could solve $Tf = \cos(t)$ in W .
 The general solution of $Tf = 0$ was $\alpha \cos(t) + \beta \sin(t)$
 ($\Rightarrow \dim(\text{Ker}(T)) = 2$). Can we solve $Tf = g$ in W
 for any $g \in W$? NO. Only for those g in
 $\text{Im}(T)$, which has dimension $\dim(\text{Im}(T)) = 4 - \dim(\text{Ker}(T))$
 $= 4 - 2 = 2$. //