Lecture 21: Change of Basis

A few lectures back we discussed writing vectors $\vec{v} \in V$ with respect to a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for V:

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \longleftrightarrow \vec{x} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n .$$

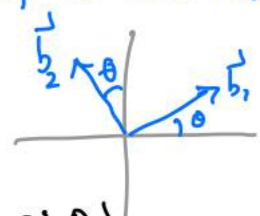
When $V = \mathbb{R}^n$, we had the change-of-basis matrix

$$P_{\mathcal{B}} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix}$$

and the relation

$$P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \vec{x} .$$

Ex 1/ $V = \mathbb{R}^2$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Find coordinates of $\vec{x}$ with respect to the rotated axes $\mathcal{B} = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}$



Write $P_{\mathcal{B}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so $P_{\mathcal{B}}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (why?).

Then $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}$. //

Ex 2/ Let V be some 2-dimensional vector space with bases

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \text{ and } \mathcal{C} = \{\vec{c}_1, \vec{c}_2\}.$$

Suppose we're given $\vec{b}_1 = -\vec{c}_1 + 4\vec{c}_2$ & $\vec{b}_2 = 3\vec{c}_1 - 6\vec{c}_2$.
If $\vec{x} = 2\vec{b}_1 + 3\vec{b}_2$, what is $[\vec{x}]_{\mathcal{C}}$?

Write $\vec{x} = 2(-\vec{c}_1 + 4\vec{c}_2) + 3(3\vec{c}_1 - 6\vec{c}_2)$

$$= (-1\cdot 2 + 3\cdot 3)\vec{c}_1 + (4\cdot 2 - 6\cdot 3)\vec{c}_2 = 7\vec{c}_1 - 10\vec{c}_2 .$$

Notice that this also can be rewritten as...

$$[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} -1 & 3 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow \\ [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \\ \downarrow & \downarrow \end{pmatrix}\begin{pmatrix} \uparrow \\ [\vec{x}]_{\mathcal{B}} \\ \downarrow \end{pmatrix} =: \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$$

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More generally, if $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ and $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ are bases for V, set

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} := \begin{pmatrix} \uparrow & & \uparrow \\ [\vec{b}_1]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \\ \downarrow & & \downarrow \end{pmatrix}$$

So that if $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, then

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}} = \alpha_1 [\vec{b}_1]_{\mathcal{C}} + \cdots + \alpha_n [\vec{b}_n]_{\mathcal{C}} = [\alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n]_{\mathcal{C}} = [\vec{x}]_{\mathcal{C}} .$$

Properties: Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} [\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\vec{x}]_{\mathcal{C}}$ for all $\vec{x}$, and so

(i) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Given another basis $\mathcal{A}$, we have also

(ii) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot \underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{A}}{P}$.

Ex 3/ Rewrite $f(t) = 5 - 6t + 3t^2 + t^3$ in powers of $(t-1)$.

In $\mathbb{P}_3$, write $\mathcal{B} = \{1, t, t^2, t^3\}$ and $\mathcal{C} = \{1, t-1, (t-1)^2, (t-1)^3\}$,

so that $[f]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -6 \\ 3 \\ 1 \end{pmatrix}$. Now $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{pmatrix} \uparrow & & \uparrow \\ [\vec{c}_1]_{\mathcal{B}} & \cdots & [\vec{c}_4]_{\mathcal{B}} \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\xRightarrow[\text{using RREF}]{\text{take inverse}}$ $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and thus $[f]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[f]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 3 \\ 6 \\ 1 \end{pmatrix}$.

That is, $f(t) = 3 + 3(t-1) + 6(t-1)^2 + (t-1)^3$. //

Now let $V = \mathbb{R}^n$, $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ (standard basis), and $\mathcal{B}$ & $\mathcal{C}$ other bases.

Properties: Since $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x} = [\vec{x}]_{\mathcal{E}}$, really

- $P_{\mathcal{B}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$, and $P_{\mathcal{B}}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$.

Combining this with (ii) above,

- $P_{\mathcal{C}}^{-1} P_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P} \cdot \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

Think: $P_{\mathcal{B}}$ sends $[\vec{x}]_{\mathcal{B}}$ to $\vec{x}$, then $P_{\mathcal{C}}^{-1}$ sends $\vec{x}$ to $[\vec{x}]_{\mathcal{C}}$.

Ex 4/ In $\mathbb{R}^2$, find the change-of-basis matrix from
$\mathcal{C} = \left\{ \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ to $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}$.

$$P_{\mathcal{B}} = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix} \Rightarrow P_{\mathcal{B}}^{-1} = \frac{1}{1\cdot 2 - (-2)(-5)} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} = -\frac{1}{8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix}$$

$$\Rightarrow \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = P_{\mathcal{B}}^{-1} P_{\mathcal{C}} = \frac{-1}{8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -3 & \frac{1}{2} \\ -5 & \frac{7}{4} \end{pmatrix}$$

$[\vec{c}_1]_{\mathcal{B}}$ $[\vec{c}_2]_{\mathcal{B}}$

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Here is another way to proceed. Consider that we want to find the coordinate vectors of $\vec{c}_1, \vec{c}_2$ wrt $\mathcal{B}$. That is, we want to solve

$$P_{\mathcal{B}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{c}_1 \quad , \quad P_{\mathcal{B}}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \vec{c}_2 ,$$

for which one would ordinarily row-reduce $[P_{\mathcal{B}} \mid \vec{c}_1]$, $[P_{\mathcal{B}} \mid \vec{c}_2]$.

An even better way to think of this is: adjoining both $\vec{c}_1$ & $\vec{c}_2$ to $P_{\mathcal{B}}$, we get

the desired $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

$$\left[\, P_{\mathcal{B}} \mid P_{\mathcal{C}} \,\right] = \left[\, \vec{b}_1 \ \vec{b}_2 \mid \vec{c}_1 \ \vec{c}_2 \,\right] \xrightarrow[\text{RREF}]{} \left[\, I_2 \mid M \,\right].$$

To see what M is, notice that the row-reduction has to multiply $P_{\mathcal{B}}$ by $P_{\mathcal{B}}^{-1}$ to get I_2. So

$$M = P_{\mathcal{B}}^{-1} P_{\mathcal{C}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} .$$

Ex 4 (cont'd) /

$$\begin{array}{c} \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & 1 \end{array}\right] \\ \begin{array}{cccc} \vec{b}_1 & \vec{b}_2 & \vec{c}_1 & \vec{c}_2 \end{array} \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -14 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 7/4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1/2 \\ 0 & 1 & -5 & 7/4 \end{array}\right]$$

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