Lecture 22: Some Applications

Recurrence relations

Given an $n \times n$ matrix A, one is sometimes interested in studying sequences of vectors $\{\vec{x}_k\} \subset \mathbb{R}^n$ ($k \in \mathbb{Z}$ or just $\mathbb{Z}_{\geq 0}$) satisfying the first-order linear recurrence relation (or "difference equation")

$$(*) \qquad \vec{x}_{k+1} = A\vec{x}_k .$$

Typically one is interested in the long-term behavior (e.g., convergence) of $\vec{x}_k$. Often $(*)$ arises from a higher-order difference equation, as in the following example.

Ex 1/ Consider the set $\mathcal{S}$ of sequences $\{y_k\}_{k\in\mathbb{Z}}$ satisfying

$$(\dagger) \qquad y_{k+2} + a y_{k+1} + b y_k = 0 \qquad (b \neq 0).$$

(i) Show that this is a subspace of

$\mathbb{S} :=$ vector space of all sequences $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$

of dimension 2.

(ii) Write it in the form $(*)$.

(i) Consider the linear transformation

$$T : \mathbb{S} \longrightarrow \mathbb{S}$$
$$\{y_k\}_{k\in\mathbb{Z}} \longmapsto \{w_k\}_{k\in\mathbb{Z}}, \text{ where } w_k := y_{k+2} + a y_{k+1} + b y_k .$$

$\mathcal{S} = \{\text{Solutions to } (\dagger)\} = \ker(T)$, which we know is a subspace.

Next, look at the linear transformation

$$R : \mathcal{S} \longrightarrow \mathbb{R}^2$$
$$\{y_k\}_{k\in\mathbb{Z}} \longmapsto \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

If $y_0 = 0 = y_1$, then using $(\dagger)$ in the form $y_{k+2} = -a y_{k+1} - b y_k$ gives $y_2 = 0$, then $y_3 = 0$, etc.; while using $(\dagger)$ in the form $y_k = \dfrac{y_{k+2} + a y_{k+1}}{-b}$ gives $y_{-1} = 0$, then $y_{-2} = 0$, etc. So $\{y_k\}$ is identically 0. This means R is <u>1-to-1</u>.

If y_0 & y_1 are any values, then once again $y_{k+2} = -a y_{k+1} - b y_k$ and $y_k = \dfrac{y_{k+2} + a y_{k+1}}{-b}$ yield a solution to $(\dagger)$ (compatible with y_0 & y_1). So R is <u>onto</u>.

Therefore R is an <u>isomorphism</u>, and $\dim \mathcal{S} = \dim \mathbb{R}^2 = 2$.

(ii) $$\begin{pmatrix} y_{k+1} \\ y_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix}$$
$$\vec{x}_{k+1} = A \cdot \vec{x}_k .$$

More generally, one has the

<u>Theorem</u>: (1) Any difference equation of the form

"order n inhomogeneous difference equation"

$$\begin{cases} y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \\ y_0, y_1, \ldots, y_{n-1} \text{ given} \qquad (a_n \neq 0) \end{cases}$$

has a unique solution in $\mathbb{S}$.

(2) The set of all solutions to

"order n homogeneous difference eqn."

$$\left\{ y_{k+n} + a_1 y_{k+n-1} + \cdots + a_n y_k = 0 \qquad (a_n \neq 0) \right.$$

is an n-dimensional vector subspace of $\mathbb{S}$.

One way to think of this: $\mathbb{S}$ = space of "signals"

"Continuous" world $\xrightarrow{\text{sample}}$ signal $\{y_k\}$ $\xrightarrow{\text{filter}}$ filtered signal $\{z_k\}$

The solutions of the homogeneous difference equation are then the ones that get filtered <u>out</u>.

Now let's actually solve one of these.

Ex 2/ Find a basis of the solution space of

$$y_{k+3} - 2y_{k+2} - y_{k+1} + 2y_k = 0.$$

We <u>could</u> say: since $\mathbb{S} \xrightarrow{\cong} \mathbb{R}^3$, $\{y_k\} \longmapsto \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$, just set $\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;

and take the 3 corresponding sequences $\{y_k\}$. But a general formula for the y_k's in this setup may be hard.

Better approach: (1) If the <u>auxiliary equation</u> $t^3 - 2t^2 - t + 2 = 0$

has a real root r, then $y_k = r^k$ is a solution. Why?
Substitute in r^k for y_k: $\quad r^{k+3} - 2r^{k+2} - r^{k+1} + 2r^k = 0 \ (\forall k)$

$$\Updownarrow$$

$$r^3 - 2r^2 - r + 2 = 0.$$

(2) If the auxiliary equation has <u>distinct</u> real roots r_1, r_2, r_3, then $\{r_1^k\}, \{r_2^k\}, \{r_3^k\}$ are independent in $\mathbb{S}$. Why?
Again, it's enough to check that the 3 values of $\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$ you get are independent, so take

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ r_1 & r_2-r_1 & r_3-r_1 \\ r_1^2 & r_2^2-r_1^2 & r_3^2-r_1^2 \end{vmatrix} = (r_2-r_1)(r_3^2-r_1^2) - (r_3-r_1)(r_2^2-r_1^2)$$

$$= (r_2-r_1)(r_3-r_1)(r_3-r_2) \neq 0.$$

(This is nonzero since r_1, r_2, r_3 are distinct.)

Solution: Solve the auxiliary equation $\rightsquigarrow (t+1)(t-1)(t-2) = 0$

$\Longrightarrow \{(-1)^k\}, \{1^k\}$, and $\{2^k\}$ are the desired basis. //

$\boxed{\text{Markov chains}}$

In probability theory, the knowledge of previous experimental outcomes can influence predictions for future experiments — this is called <u>conditional probability</u>.

If you repeatedly carry out such an experiment or measurement, then you get a sequence of probability distributions and can ask where they go in the long run.

Here the probability distribution will be represented by a <u>state vector</u> whose entries sum to 1, say

$$\vec{x}_k = \begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix} \quad \text{where } P_k(E) := \text{probability of event } E \text{ occurring at the } k^{\text{th}} \text{ step.}$$

We have a matrix of conditional probabilities

$$P = \begin{pmatrix} P(A|A) & P(A|B) \\ P(B|A) & P(B|B) \end{pmatrix} \quad \text{where } P(E|F) := \text{probability event } E \text{ occurs at next step when } F \text{ has occurred at current step}$$

in which the columns sum to 1 —

such a matrix is called <u>stochastic</u>. (This ensures that the entries of $\vec{x}_{k+1}$ sum to 1 when $\vec{x}_k$'s do: $\underbrace{(1 \ \ 1)}_{1\times 2 \text{ matrix}} \cdot \vec{x}_{k+1}$
$= (1 \ \ 1) \cdot P \cdot \vec{x}_k = (1 \ \ 1)\left(\vec{p}_1 \ \ \vec{p}_2\right)\vec{x}_k \underset{\text{colums of } P \text{ sum to } 1}{=} (1 \ \ 1)\,\vec{x}_k = 1.$)

So then

$$\underbrace{\begin{pmatrix} P_{k+1}(A) \\ P_{k+1}(B) \end{pmatrix}}_{\vec{x}_{k+1}} = \underbrace{\begin{pmatrix} P(A|A) & P(A|B) \\ P(B|A) & P(B|B) \end{pmatrix}}_{P} \cdot \underbrace{\begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix}}_{\vec{x}_k},$$

and the sequence

$$\vec{x}_0, \ \vec{x}_1, \ \vec{x}_2, \ \vec{x}_3, \ \ldots$$

is called a <u>Markov chain</u>.

Ex 3 (Winter in MO) / There are 3 kinds of weather:

W = warm
T = tornadoes
C = cold/clear

e.g. a warm day might have a $\begin{cases} 25\% \text{ chance of being followed by } W \\ 50\% \text{ " " " " " } T \\ 25\% \text{ " " " " " } C \end{cases}$

So say our stochastic matrix is

$$P = \begin{array}{cc} \begin{array}{ccc} W & T & C \end{array} & \\ \begin{pmatrix} 1/4 & 0 & 1/4 \\ 1/2 & 0 & 0 \\ 1/4 & 1 & 3/4 \end{pmatrix} & \begin{array}{c} W \\ T \\ C \end{array} \end{array}$$

and that today is cold (sort of). Then what is the probability of a tornado the day after tomorrow?

$$P^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} 1/4 \\ 0 \\ 3/4 \end{pmatrix} = \begin{pmatrix} 1/4 \\ \boxed{1/8} \\ 5/8 \end{pmatrix} \rightsquigarrow 12.5\% \text{ chance.}$$

What is the probability of a tornado on an "average day in the long run"?

Rather than try to compute $P^N \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for N big, what we're really after is a vector $\vec{x}$ with the property that

$$P\vec{x} = \vec{x},$$

i.e. a steady-state vector. (There is a unique such vector if some power of P — in this case, P^3 — has all positive entries.) To find $\vec{x}$, write

$$\vec{0} = P\vec{x} - \vec{x} = (P - I_3)\vec{x}.$$

That is, we are after the null space of

$$P - I_3 = \begin{bmatrix} -3/4 & 0 & 1/4 \\ 1/2 & -1 & 0 \\ 1/4 & 1 & -1/4 \end{bmatrix} \xrightarrow{\times 2} \begin{bmatrix} 1 & -2 & 0 \\ -3/4 & 0 & 1/4 \\ 1/4 & 1 & -1/4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & -3/2 & 1/4 \\ \cancel{0} & \cancel{3/2} & \cancel{-1/4} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}.$$

So a solution is $\begin{pmatrix} 1/3 \\ 1/6 \\ 1 \end{pmatrix}$ — but the sum of its entries is $3/2$. To get a state vector, divide by this:

$$\vec{x} = \frac{2}{3}\begin{pmatrix} 1/3 \\ 1/6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/9 \\ \boxed{1/9} \\ 2/3 \end{pmatrix} \rightsquigarrow \underline{\text{Answer}}: \frac{1}{9}. \quad //$$

More generally, our state vectors might represent the distribution of the population or vote (over different locations resp. candidates) — see the examples in the text.