Lecture 23 : Eigenstuff

An Example (population dynamics)

We begin with a linear system of difference equations
(or "discrete dynamical system")

$$W(\tau+1) = 0.86 \cdot W(\tau) + 0.08 \cdot S(\tau)$$

$$S(\tau+1) = -0.12 \cdot W(\tau) + 1.14 \cdot S(\tau)$$

describing the wolf & sheep populations over time (measured in months). Set

$$\vec{x}(\tau) = \begin{pmatrix} W(\tau) \\ S(\tau) \end{pmatrix}, \qquad A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix},$$

so that

$$\vec{x}(\tau+1) = A \cdot \vec{x}(\tau).$$

To understand the system's long-term behavior, we need to compute

$$\vec{x}(\tau) = A^\tau \vec{x}_0 = \underbrace{A \cdot \cdots \cdot A}_{\tau \text{ times}} \begin{pmatrix} W_0 \\ S_0 \end{pmatrix}$$

$\leftarrow$ starting values at time $\tau = 0$

for very large τ.

- Consider the initial state vector $W_0 = 100$, $S_0 = 300$

$$\vec{x}(1) = A \vec{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \end{pmatrix} = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1 \cdot \vec{x}_0$$

Then $\vec{x}(\tau) = A^\tau \vec{x}_0 = (1.1)^\tau \vec{x}_0 \implies$ populations grow together.

- Consider the initial state vector $W_0 = 200$, $S_0 = 100$.

$$\vec{x}(1) = A\vec{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}\begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 180 \\ 90 \end{pmatrix} = 0.9\,\vec{x}_0$$

and $\vec{x}(\tau) = A^\tau \vec{x}_0 = (0.9)^\tau \vec{x}_0 \implies$ populations shrink together: too many wolves eating too few sheep.

Put $\vec{v}_1 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$ $\to$ "eigenvectors"

$\qquad \lambda_1 = 1.1 \qquad, \quad \lambda_2 = 0.9.$ $\to$ "eigenvalues"

Now, in your backyard, $\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$. So

$$\vec{x}(1) = A\vec{x}_0 = \begin{pmatrix} 940 \\ 1020 \end{pmatrix},$$

and we aren't in one of the neat cases above. But we can still use those cases to understand what is going on in the long term: break $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 \implies$

$$\vec{x}(1) = A\vec{x}_0 = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = c_1(1.1)\vec{v}_1 + c_2(0.9)\vec{v}_2$$

and $\vec{x}(\tau) = \underline{c_1(1.1)^\tau \vec{v}_1} + \underline{c_2(0.9)^\tau \vec{v}_2};$
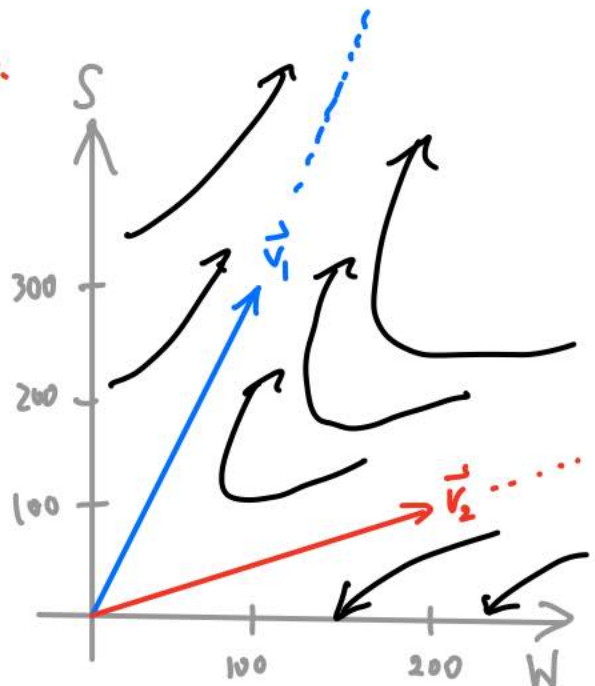
grows by 10% every month   shrinks by 10% every month

So in case 4

$$\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2\begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4\begin{pmatrix} 200 \\ 100 \end{pmatrix} = 2\vec{v}_1 + 4\vec{v}_2,$$

we have

$$\vec{x}(\tau) = 2(1.1)^\tau\begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4(0.9)^\tau\begin{pmatrix} 200 \\ 100 \end{pmatrix}.$$

The flow lines at right depict the path taken by $\vec{x}(\tau)$. Clearly we need to get the sheep/wolf population ratio below 1:2 ...

A little theory

Let A be an $n \times n$ matrix.

Definition: A nonzero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of A if

$$A\vec{v} = \lambda \vec{v}$$

for some scalar λ. In this case, λ is called an eigenvalue of A, and $\vec{v}$ is called an eigenvector (of A) with eigenvalue λ.

Let

$$E_\lambda := \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda \vec{v} \}$$

be the set of all eigenvectors with eigenvalue λ (together with the zero vector). This is the eigenspace associated to λ. Now we make the fundamental

Observation: $A\vec{v} = \lambda \vec{v} \iff (A - \lambda I_n)\vec{v} = \vec{0}$,

i.e. $E_\lambda = \text{Nul}(A - \lambda I_n)$, and is

therefore a subspace of $\mathbb{R}^n$! In addition,

λ is an eigenvalue of $A \iff \text{Nul}(A - \lambda I_n) \neq \{\vec{0}\}$

$\iff A - \lambda I_n$ is not invertible

$\iff \det(A - \lambda I_n) = 0$,

proving

Theorem 1: The eigenvalues of A are the solutions to the characteristic equation
$$\det(A - \lambda I_n) = 0.$$

Suppose A is upper triangular, say
$$A = \begin{pmatrix} \alpha_1 & * & * \\ 0 & \alpha_2 & * \\ 0 & 0 & \alpha_3 \end{pmatrix}. \qquad [* = \text{anything}]$$

Then $A - \lambda I_3 = \begin{pmatrix} \alpha_1 - \lambda & * & * \\ 0 & \alpha_2 - \lambda & * \\ 0 & 0 & \alpha_3 - \lambda \end{pmatrix}$

$$\Rightarrow \det(A - \lambda I_3) = (\alpha_1 - \lambda)(\alpha_2 - \lambda)(\alpha_3 - \lambda).$$

Corollary: The eigenvalues of an upper triangular matrix are the diagonal entries.

Another Example

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find all the eigenvalues and bases/dimension for each eigenspace.

(A) First solve the characteristic equation

$$\det(A - \lambda \mathbb{I}_3) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix}$$

subtract / add

$$= \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2-\lambda \\ 0 & \lambda & 0 \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}$$

now do Laplace

$$= -\lambda \{ (\lambda-1)(\lambda-2) - 2\cdot 1 \} = -\lambda \{ \lambda^2 - 3\lambda + 2 - 2 \}$$

$$= -\lambda^2 (\lambda - 3)$$

$\implies$ eigenvalues are 0 & 3.

(B) Then compute bases for E_0 & E_3, viewed as null spaces

• For $E_3 = \text{Nul}(A - 3\mathbb{I}_3) = \text{Nul}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

row-reduce:
$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 free

$\implies$ basis for E_3 is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \implies \dim E_3 = 1$.

• For $E_0 = \text{Nul}(A - 0\mathbb{I}_3) = \text{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

row-reduce:
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3 free

$\implies$ basis for E_0 is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\implies \dim E_0 = 2$.

And in fact, putting the two bases together gives a basis for $\mathbb{R}^3$. This doesn't always happen. In the next lecture we'll see why it does happen in some cases (including here).

Some problems

(1) Given 2 eigenvectors of A, is their sum an eigenvector?

(2) In each case, is $\vec{v}$ an eigenvector of A? If so, find the eigenvalue:

(a) $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $A = \begin{pmatrix} -3 & 1 \\ -3 & 8 \end{pmatrix}$; (b) $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

(3) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$.