Lecture 24 : Diagonalizing Matrices

Recall that when $\vec{v} \in \mathbb{R}^n$ is nonzero, $\lambda \in \mathbb{R}$, and

$$A\vec{v} = \lambda \cdot \vec{v}, \qquad (A = n \times n \text{ matrix})$$

λ [resp. $\vec{v}$] is an eigenvalue [resp. eigenvector] of A.

- to find eigenvalues: solve $\det(A - \lambda I_n) = 0$
- to find eigenvectors: for each eigenvalue λ_0, find (a basis for)
$$E_{\lambda_0} = \mathrm{Nul}(A - \lambda_0 I_n)$$
by row-reduction. Its dimension is $n - \mathrm{rk}(A - \lambda_0 I_n)$, by R+N.
- to check if $\vec{v}_0$ is an eigenvector: apply A to $\vec{v}$
- to check if λ_0 is an eigenvalue: see if $\mathrm{rk}(A - \lambda_0 I_n) < n$ by using row-reduction.
- the eigenvalues of an upper or lower-triangular matrix are the diagonal entries.

Now by the Fundamental Theorem of Algebra, the characteristic polynomial factors

$$(*) \qquad \det(A - \lambda I_n) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

where in general the $\{\lambda_i\}$ may be non-real (i.e. complex numbers) and may not be distinct. Assume for now that they <u>are</u> real.

Definition: The multiplicity of an eigenvalue of A is the number of times it appears in (*). (If all multiplicities are 1, then A has n distinct eigenvalues.)

Lemma: If $\vec{v}_1, \dots, \vec{v}_k$ are k eigenvectors of A with distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then they are linearly independent.

Proof: Use induction: this is clear for $k=1$, since $\vec{v}_1 \neq \vec{0}$ by definition. Assume it holds for $k-1$ eigenvectors with distinct eigenvalues, i.e. that $\vec{v}_1, \dots, \vec{v}_{k-1}$ are independent, and let $\vec{v}_k$ be an eigenvector with a "new" eigenvalue.

Suppose $\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$. (We must show the c_i's are all 0.) On one hand (multiplying by λ_k)

(1) $$\vec{0} = c_1\lambda_k\vec{v}_1 + \cdots + c_k\lambda_k\vec{v}_k.$$

On the other hand (applying A)

(2) $$\vec{0} = c_1 A\vec{v}_1 + \cdots + c_k A\vec{v}_k = c_1\lambda_1\vec{v}_1 + \cdots + c_k\lambda_k\vec{v}_k.$$

Subtracting (1) - (2) gives

$$\vec{0} = c_1\underbrace{(\lambda_k - \lambda_1)}_{\neq 0}\vec{v}_1 + \cdots + c_{k-1}\underbrace{(\lambda_k - \lambda_{k-1})}_{\neq 0}\vec{v}_{k-1} + c_k\cancelto{0}{(\lambda_k - \lambda_k)}\vec{v}_k$$

$\Longrightarrow$ $c_1 = \cdots = c_{k-1} = 0$ (since $\vec{v}_1, \dots, \vec{v}_{k-1}$ are L.I.).

But then the original equation reads $\vec{0} = c_k\vec{v}_k \Rightarrow c_k = 0$. $\square$

Theorem: If the eigenvalues of A are distinct (and real), then $\underbrace{\text{a basis of } \mathbb{R}^n \text{ consisting of eigenvectors of } A}_{\text{"A-eigenbasis"}}$ exists.

Proof: For each of the n eigenvalues, there's an eigenvector. Apply the Lemma. $\square$

Remark: The existence of an A-eigenbasis is crucial for being able to write a given vector as a sum of eigenvectors of A, as part of solving systems of difference / differential equations, etc.

Given an eigenbasis $\vec{v}_1, \dots, \vec{v}_n$, write $P = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$ and compute $AP = \begin{pmatrix} \uparrow & & \uparrow \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$; assemble the eigenvalues into a diagonal matrix $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, so that $PD = \begin{pmatrix} \uparrow & & \uparrow \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$. Hence $AP = PD$ and

$$A = P \cdot D \cdot P^{-1} \tag{$\dagger$}$$

(equivalently $P^{-1}AP = D$). We have **diagonalized** A.

Corollary 1: A matrix with n distinct eigenvalues can be diagonalized (or "is **diagonalizable**").

Ex 1/ Diagonalise $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. From lecture 23,

A has

$$\begin{cases} \text{eigenvalues: } \underbrace{0, 0}_{\text{multiplicity } 2}, 3 \\ \qquad\qquad \updownarrow \\ \text{eigenvectors: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{cases}$$

multiplicity 2 — recall characteristic polynomial was $-\lambda^2(\lambda - 3)$.

So Lemma doesn't apply — in this case, check independence.

eigenbasis

(must be in the same order)

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \boxed{0} & 0 & 0 \\ 0 & \boxed{0} & 0 \\ 0 & 0 & \boxed{3} \end{pmatrix}$$

Therefore $A = \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{P^{-1}}$.

So you don't need to have distinct eigenvalues in order to diagonalize. //

Ex 2/ Is $A = \begin{pmatrix} 5 & -8 & -21 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$ diagonalizable? If so, do it.

eigenvalues: $\overset{\lambda_1}{5}, \overset{\lambda_2}{0}, \overset{\lambda_3}{-2}$ (upper triangular) $\to$ distinct $\Rightarrow$ diagonalizable.

eigenvectors: row-reduce (if necessary) to find vector in null spaces of

$$A - 5I = \begin{pmatrix} 0 & -8 & -21 \\ & -5 & 7 \\ & & -7 \end{pmatrix} \longrightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A - 0I = \begin{pmatrix} 5 & -8 & -21 \\ & 0 & 7 \\ & & -2 \end{pmatrix} \longrightarrow \vec{v}_2 = \begin{pmatrix} 8 \\ 5 \\ 0 \end{pmatrix}$$

$$A + 2I = \begin{pmatrix} 7 & -8 & -21 \\ & 2 & 7 \\ & & 0 \end{pmatrix} \longrightarrow \vec{v}_3 = \begin{pmatrix} -1 \\ -7/2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad D = \begin{pmatrix} 5 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 8 & -1 \\ & 5 & -7/2 \\ & & 1 \end{pmatrix}$$

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Ex 3/ What about $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$? (lower triangular)

eigenvalues: $2, 3$ w/multiplicity 2

Find eigenvectors: $A - 2I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ⤳ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ spans E_2.

$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ⤳ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ spans E_3.

Up to scale, there are only 2 eigenvectors. So there's no A-eigenbasis of $\mathbb{R}^3$, and A isn't diagonalizable. //

Applications

(1) The determinant of a diagonalizable $n \times n$ matrix is the product of the n (not necessarily distinct) eigenvalues:

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n .$$

Why? $\det A = \det(PDP^{-1}) = \det P \cdot \det D \cdot \det P^{-1}$

$$= \det D = \det \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \prod_{i=1}^{n} \lambda_i .$$

(In fact, this is still true even if A isn't diagonalizable: we still have $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$ by $(*)$, with constant term $\lambda_1 \cdots \lambda_n$. But another way to compute this constant term is by setting $\lambda = 0$, giving $\det(A - 0I) = \det(A)$. So $\det(A) = \lambda_1 \cdots \lambda_n$.)

(2) Compute $\underbrace{\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}}_{A}^{10}$ by diagonalizing A:

$A = PDP$ where $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$, $P = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$

$$\Rightarrow A^{10} = (PDP^{-1})^{10} = PD\cancel{P^{-1} \cdot P}D\cancel{P^{-1} \cdot P}D\cancel{P^{-1} \cdot \ldots \cdot P}DP^{-1}$$

$$= PD^{10}P^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} (-2)^{10} & 0 \\ 0 & 5^{10} \end{pmatrix} \begin{pmatrix} 4/7 & -3/7 \\ -1/7 & -1/7 \end{pmatrix}$$

$$= \frac{1}{7} \left(\begin{array}{c|c} -4 \cdot 2^{10} - 3 \cdot 5^{10} & 3 \cdot 2^{10} - 3 \cdot 5^{10} \\ \hline 4 \cdot 2^{10} - 4 \cdot 5^{10} & -3 \cdot 2^{10} - 4 \cdot 5^{10} \end{array}\right).$$

(3) Stochastic matrices

Let A be an $n \times n$ matrix with all positive entries, whose columns sum to 1. (This is called a regular stochastic matrix.)

- A has a steady-state vector, i.e. eigenvector with eigenvalue 1:

Since columns of A sum to 1, columns of $A - I_n$ sum to 0 hence belong to an $(n-1)$-dim'l subspace of $\mathbb{R}^n$ and cannot all be independent $\Longrightarrow$ rank$(A - I_n) \leq n-1 \Longrightarrow$ nullity$(A - I_n) \geq 1$. In fact, by a careful examination of $A - I$ ⇝ picture: $\begin{pmatrix} - & + & + & \cdots \\ + & - & + & \cdots \\ + & + & - & \cdots \\ \vdots & & & \ddots \end{pmatrix}$ ⇝ one can deduce that $\dim E_1$ $(=$ nullity$(A - I_n))$ $= 1$, so the steady-state vector is unique (provided we scale it to have its entries sum to 1). That is, the multiplicity of the eigenvalue 1, is 1.

- Any eigenvector with a different eigenvalue than 1
  - (a) must lie in the plane $x_1 + \cdots + x_n = 0$
  - (b) must have eigenvalue $\in (-1, 1)$

(b) is important for dynamical systems / Markov chains — if the initial state is $\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots$ and $\vec{v}_1$ = steady state vector, then

$$\vec{x}(t) = A^t \vec{x}_0 = c_1 \vec{v}_1 + c_2 \underbrace{\lambda_2^t}_{\to 0 \text{ since } |\lambda_2| < 1} \vec{v}_2 + \cdots \xrightarrow[t \to \infty]{\text{limit as}} c_1 \vec{v}_1$$

EXAMPLE:

- 2×2 matrices: $A = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix} \Longrightarrow A - \lambda I_2 = \begin{pmatrix} a-\lambda & 1-b \\ 1-a & b-\lambda \end{pmatrix}$

$$\Longrightarrow \det(A - \lambda I_2) = \lambda^2 - (a+b)\lambda + (a+b-1) = 0$$

$$\Longrightarrow \lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(a+b-1)}}{2} = \begin{cases} 1 \\ a+b-1 \end{cases}$$

Since $0 < a+b < 2$, $-1 < a+b-1 < 1$ as desired.