Lecture 27: Complex Eigenvalues

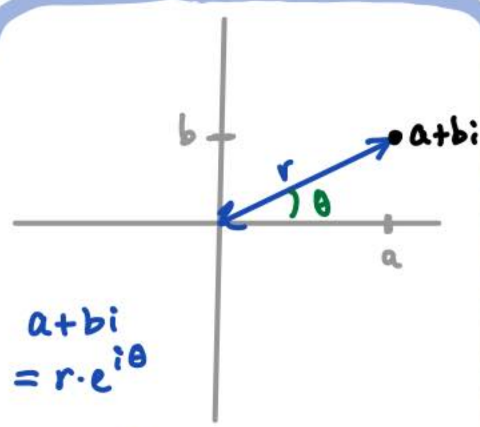
In Lecture 25 we began to discuss complex† eigenvalues, eigenvectors, and diagonalization — for real 2×2 matrices M. First, the characteristic polynomial & quadratic equation showed us that we will have complex eigenvalues/etc. precisely when

$$(tr(M))^2 < 4\det(M).$$

A special case was that of "rotation-dilation matrices"

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r \cdot R_\theta = r \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = \arctan b/a$ and $r = \sqrt{a^2+b^2}$.



$a+bi$
$= r\cdot e^{i\theta}$

polar form of complex #'s

Notice that $tr(M) = 2a$ and $\det(M) = a^2+b^2 \implies$

$$tr(M)^2 = 4a^2 < 4a^2+4b^2 = \det(M)$$

as long as $b \neq 0$. The eigenvectors and eigenvalues of this M were discovered to be

$$\left.\begin{array}{ll} \vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, & \lambda_1 = a+bi \\ \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}, & \lambda_2 = a-bi \end{array}\right\} \rightsquigarrow N := \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix},\ D = \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix},$$

$$N^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

† If you need to review complex numbers, see Appendix B in your book.

and M diagonalizes:

$$(*) \qquad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = N D N^{-1}$$

Let's consider now the general case of a 2×2 matrix A with real entries but a complex eigenvalue $a-bi$ (assume $b\neq 0$) with eigenvector

$$\vec{v} = \vec{u} + i\vec{w} =: \underbrace{Re(\vec{v})}_{\text{real part}} + i\cdot\underbrace{Im(\vec{v})}_{\text{imaginary part}}$$

(both are real vectors in spite of the names)

That is,

$$A(\vec{u}+i\vec{w}) = (a-bi)(\vec{u}+i\vec{w}).$$

Apply complex conjugation (noting that A is real, so $\bar{A}=A$)

$$\overline{A}\,\overline{(\vec{u}+i\vec{w})} = \overline{(a-bi)}\,\overline{(\vec{u}+i\vec{w})}$$

$$\overset{\shortparallel}{A(\vec{u}-i\vec{w})} \qquad \overset{\shortparallel}{(a+bi)(\vec{u}-i\vec{w})}$$

$\Longrightarrow$ $\vec{u}-i\vec{w}$ is an eigenvector with eigenvalue $a+bi$.

(Since the eigenvalues are distinct, the eigenvectors must be as well: so $\vec{w}\neq\vec{0}$.) Taking

$$S = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u}-i\vec{w} & \vec{u}+i\vec{w} \\ \downarrow & \downarrow \end{pmatrix},$$

we have

$$A = SDS^{-1} = S\begin{pmatrix} a+ib & \\ & a-ib \end{pmatrix} S^{-1}$$

usual diagonalization

$$= SN^{-1}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} NS^{-1}.$$

by (*) in the form $N^{-1}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} N = D$

If we set

$$P := SN^{-1} = \frac{1}{2}\begin{pmatrix} \uparrow & \uparrow \\ \vec{u}-i\vec{w} & \vec{u}+i\vec{w} \\ \downarrow & \downarrow \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \uparrow & \uparrow \\ \left\{\begin{matrix} \vec{u}-i\vec{w} \\ +\vec{u}+i\vec{w} \end{matrix}\right\} & \left\{\begin{matrix} i(\vec{u}-i\vec{w}) \\ -i(\vec{u}+i\vec{w}) \end{matrix}\right\} \\ \downarrow & \downarrow \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \uparrow & \uparrow \\ 2\vec{u} & 2\vec{w} \\ \downarrow & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & \vec{w} \\ \downarrow & \downarrow \end{pmatrix},$$

then we see right away the following

Theorem: A is similar[††] to a rotation-dilation matrix. Explicitly, $A = P\begin{pmatrix} a & -b \\ b & a \end{pmatrix}P^{-1}$ with $P = \begin{pmatrix} \uparrow & \uparrow \\ \mathrm{Re}(\vec{v}) & \mathrm{Im}(\vec{v}) \\ \downarrow & \downarrow \end{pmatrix}$ (and $\vec{v}$ = eigenvector w/eigenvalue $a-bi$).

---

†† "A is similar to B" means that $A = SBS^{-1}$ for some invertible S.

Example / Demonstrate the Theorem in case $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Discuss what happens if we repeatedly apply A to a "state vector".

- Solving $0 = \det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)+5 = \lambda^2 - 2\lambda + 2$

yields $\lambda = 1 \pm i =: a \pm ib$, say $a = b = 1$.

- Moreover, $E_{\underbrace{1-i}_{a-bi}} = \mathrm{Nul}\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & -2+i \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\underbrace{\begin{pmatrix} 2-i \\ 1 \end{pmatrix}}_{=:\vec{v}}\right\}$

and so $\vec{u} := \mathrm{Re}\,\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ while $\vec{w} := \mathrm{Im}(\vec{v}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

yielding $P = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

- By the Theorem,

$$A = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\sqrt{2}\cdot R_{\pi/4}} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}}_{P^{-1}}.$$

- We can conceptualize this in terms of transformations:
if $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation with (standard) matrix A,
then in the basis $\mathcal{B} = \{\vec{u}, \vec{w}\}$ we have

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2} \cdot R_{\pi/4}.$$

- Now let $\vec{u}$ be a "state vector"; what can we say about $A^k \vec{u}$ in the long run? In $\mathcal{B}$-coordinates,

(Same $\vec{u}$ as above. Could take any $\vec{x}$ here; just easy to use $\vec{u}$.)

$$[A^k \vec{u}]_{\mathcal{B}} = [T^k(\vec{u})]_{\mathcal{B}} = ([T]_{\mathcal{B}})^k [\vec{u}]_{\mathcal{B}} = \sqrt{2}^k R_{\frac{k\pi}{4}} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

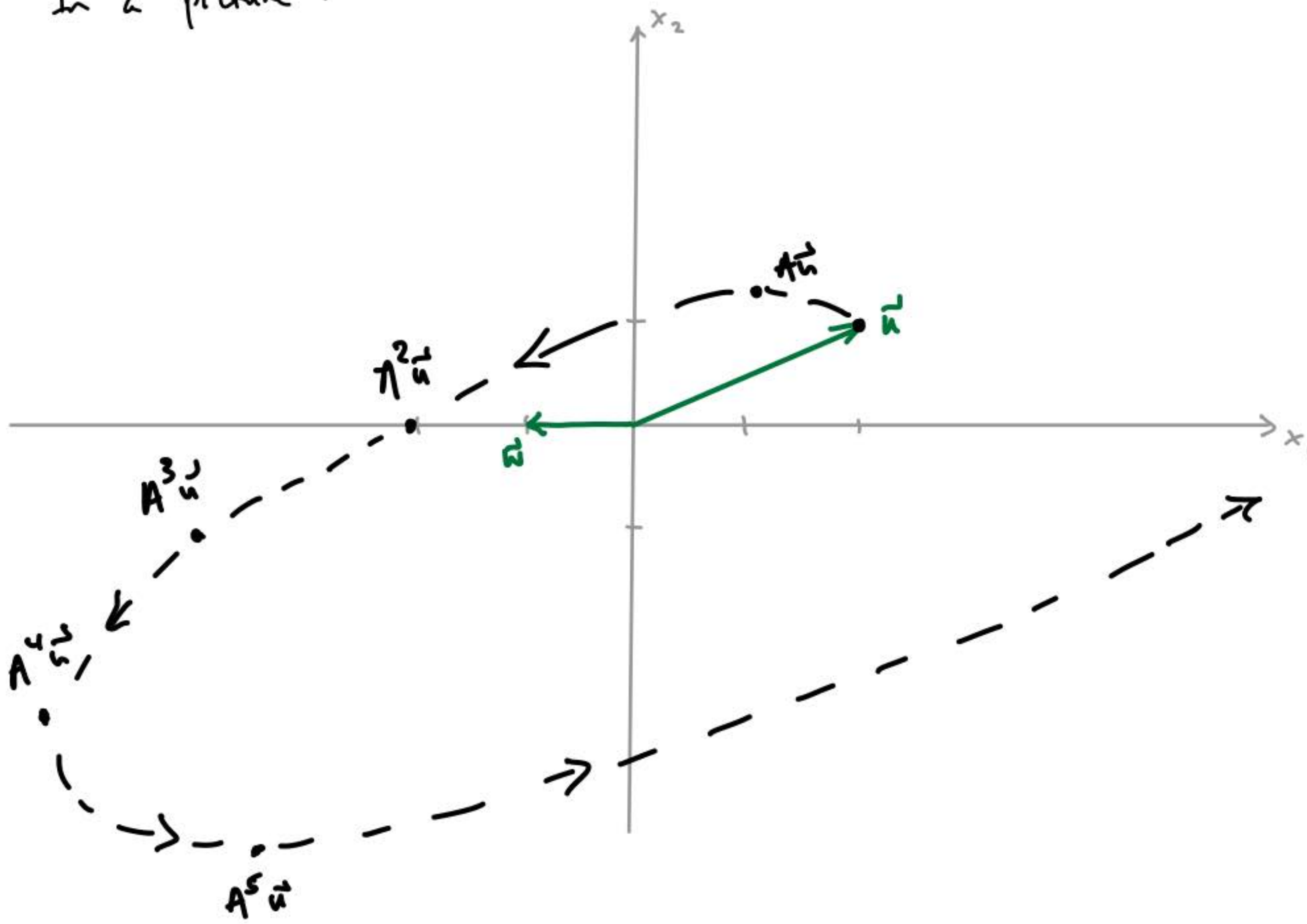
$$= \sqrt{2}^k \begin{pmatrix} \cos \frac{k\pi}{4} \\ \sin \frac{k\pi}{4} \end{pmatrix}.$$

b/c $\vec{u}$ is the first vector in the basis $\mathcal{B} = \{\vec{u}, \vec{w}\}$

- So in the standard basis

$$A^k \vec{u} = \underbrace{P}_{\begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & \vec{w} \\ \downarrow & \downarrow \end{pmatrix}} \sqrt{2}^k \begin{pmatrix} \cos(k\pi/4) \\ \sin(k\pi/4) \end{pmatrix} = \sqrt{2}^k \left\{ \cos\left(\frac{k\pi}{4}\right) \vec{u} + \sin\left(\frac{k\pi}{4}\right) \vec{w} \right\}.$$

In a picture :



This "elliptically spiralling outward" behavior is quite different from anything you see for a matrix with real eigenvalues. //