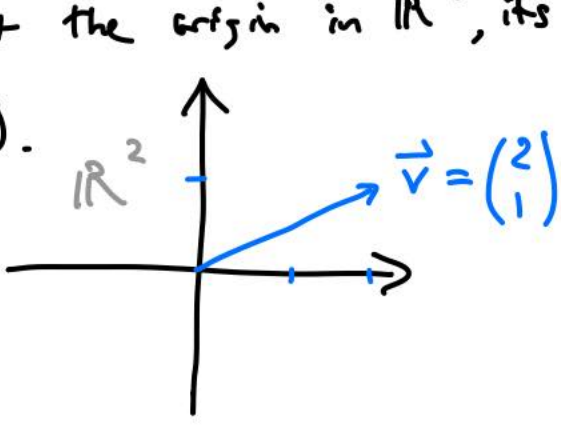
Lecture 3: Linear Combinations

until we abstractify further

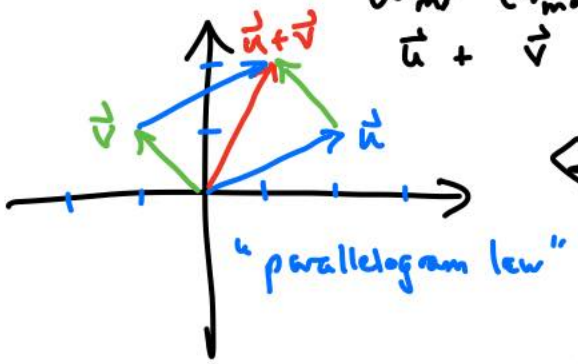
A **vector** is an m-tuple of real numbers (for some m), which we shall write sometimes in a column ($= m \times 1$ matrix) — the default — and sometimes as a row ($= 1 \times m$ matrix). **Real m-space** $\mathbb{R}^m$ consists of m-tuples (think: coordinates of a point) of real numbers, with the "origin" at $(0, \ldots, 0)$. But when we say the word "vector", we are thinking more of a directed line segment ($=$ arrow) — a "magnitude + direction" — than a point. More precisely, the vector $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ has direction & length such that if you place its tail at the origin in $\mathbb{R}^m$, its head lies at the point $(a_1, \ldots, a_m)$.



Having said all this, we will write $\vec{v} \in \mathbb{R}^m$ when we want to consider a vector in m-space.

Adding vectors: $\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_m+v_m \end{pmatrix}$

$\vec{u} \quad + \quad \vec{v}$



geometric interpretation: place vectors head to tail

Scalar multiplication: $c\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} cu_1 \\ \vdots \\ cu_m \end{pmatrix}$, for $c \in \mathbb{R}$:

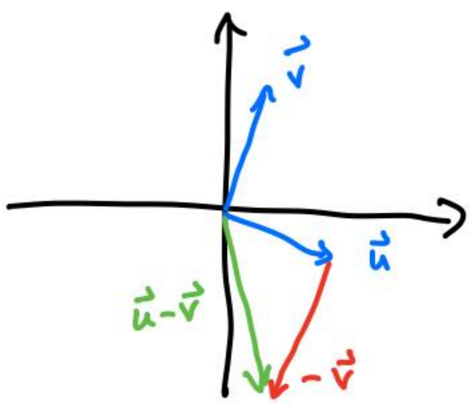
$c \cdot \vec{u}$

Corresponds to shrinking or stretching the directed segment (and reversing its direction if $c < 0$).

Definition 1: A linear combination of vectors $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ is a vector $\vec{v} \in \mathbb{R}^m$ of the form $(\vec{v} =)\ c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n$, where $c_1, \dots, c_n$ are real numbers.

Ex 1/

(in $\mathbb{R}^2$)



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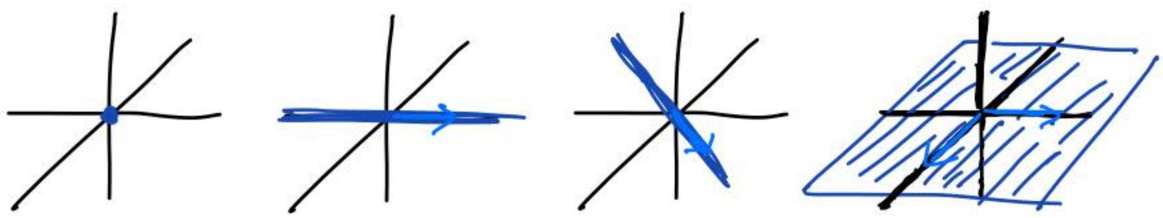
Definition 2: The span of $\vec{u}_1, \ldots, \vec{u}_n \in \mathbb{R}^m$ is the set (in $\mathbb{R}^m$) of all linear combinations of them:

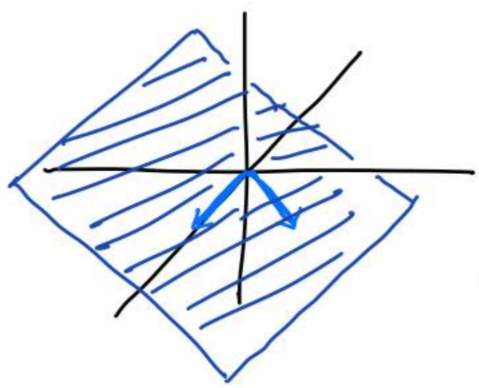
$$\operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_n\} := \left\{ c_1\vec{u}_1 + \cdots + c_n\vec{u}_n \;\middle|\; c_1, \ldots, c_n \in \mathbb{R} \right\}.$$

When drawing a span, we draw the set of points — can't draw all those vectors at once.

Ex 2/ (in $\mathbb{R}^3$) Notation: $\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $\vec{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$



$\operatorname{span}\{\vec{0}\}$ $\quad$ $\operatorname{span}\{\vec{e}_2\}$ $\quad$ $\operatorname{span}\{\vec{e}_1 + \vec{e}_2\}$ $\quad$ $\operatorname{span}\{\vec{e}_1, \vec{e}_2\}$



$\operatorname{span}\{\vec{e}_1, \vec{e}_2 - \vec{e}_3\}$

Going back to linear systems, consider the following three statements:

Statement 1: There exists a solution of the linear system

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{ m equations in n variables}$$

Statement 2: There exist $x_1, \ldots, x_n \in \mathbb{R}$ such that

$$x_1 \underbrace{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}_{\vec{u}_1} + \cdots + x_n \underbrace{\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{\vec{u}_n} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}}$$

Statement 3: $\vec{b}$ is a linear combination of $\vec{u}_1, \ldots, \vec{u}_n$;

that is, $\vec{b} \in \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_n\}$.

Ex 3/

$$\left.\begin{aligned} 2x - 3y &= 1 \\ -x + 6y &= 2 \end{aligned}\right\} \iff x\begin{pmatrix} 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \end{pmatrix}\right\} \quad //$$

These three statements are visibly equivalent. Thus: to check whether a given vector is in the span of some other vectors, amounts to solving a linear system!

Ex 4/ Is $\vec{b} = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$ in the plane generated by (i.e., span of) $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$?

If this is "Statement 3", then "Statement 1" reads

$$\left.\begin{aligned} 1x_1 - 4x_2 &= 3 \\ 0x_1 + 3x_2 &= -7 \\ -2x_1 + 8x_2 &= -3 \end{aligned}\right\} \longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 3 & -7 \\ -2 & 8 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 3 & -7 \\ 0 & 0 & 3 \end{array}\right]$$

inconsistent

Conclude that $\vec{b} \notin \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$.

Now suppose $\vec{b} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$. Same question.

$$\left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 26/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array}\right]$$

$\Longrightarrow (x_1, x_2) = \left(\frac{26}{3}, \frac{5}{3}\right)$. Check: $\frac{26}{3}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$. //

In Lecture 2, I claimed the

Theorem: Every matrix A is row-equivalent to a unique RREF matrix.

To show this, we need the following:

(*) If A is row-equivalent to B, then the rows of A are linear combinations of rows of B (and vice versa).

This is because

- the row operations — replace, swap, scale — simply replace a given row by a linear combination of rows.
- they are reversible.

To prove the Theorem, proceed in 3 steps:

1. It is enough to show that

(**) 2 row-equivalent RREF matrices U and V must be the same.

Why? By the algorithm, $A \underset{\text{row-eq.}}{\sim} \operatorname{rref}(A) =: U$.

If $A \underset{\text{row-eq.}}{\sim} V$ (also RREF), then $U \underset{\text{row-eq}}{\sim} V$.

(So if (**) is known, we get $U = V$ as desired.)

For the remaining 2 steps (which prove (**)), let U and V be 2 row-equivalent RREF matrices:

② The pivot columns of U & V are the same.

$\boxed{\text{picture of } U}$

$$\begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 \cdots \\ 0 \cdots & \cdots & \cdots 0 & 1 & * \cdots * & 0 \cdots \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots 0 & 1 \cdots \\ & & \vdots & & & \end{bmatrix}$$

column: i_1 $\quad i_2$ $\quad i_3$ $\cdots$

$\boxed{\text{picture of } V}$

$$\begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 \cdots \\ 0 \cdots & \cdots & \cdots 0 & 1 & * \cdots * & 0 \cdots \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots 0 & 1 \cdots \\ & & \vdots & & & \end{bmatrix}$$

column: j_1 $\quad j_2$ $\quad j_3$ $\cdots$

Write $\vec{u}_1, \vec{u}_2$, etc. for rows of U (viewed as vectors), same for V.

(A) By $(*)$, $\vec{u}_1 \in \text{span}\{\text{rows of } V\}$. So:

$\vec{u}_1 = a_1\vec{v}_1 + a\vec{v}_2 + \cdots \quad \Rightarrow \quad i_1 \geq j_1$.

Reversing U & V gives $j_1 \geq i_1$. So $i_1 = j_1$ (and $a_1 = 1$).

(B) Next, $\vec{u}_2 = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots \Rightarrow \vec{u}_2$ has b_1 in $(j_1 =) i_1^{\text{th}}$ coord.

$\Rightarrow b_1 = 0$

$\Rightarrow \vec{u}_2 = b_2\vec{v}_2 + b_3\vec{v}_3 + \cdots$

(and vice versa).

(C) Striking out the first rows of U and V, the remaining (still RREF!) matrices are row equivalent. Go back to (A), find $i_2 = j_2$, and repeat until there's nothing left.

③ The non-pivot columns of U & V are equal.

(D) We have $\vec{u}_1 = \vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m$ (recall $a_1 = 1$ above)

$\Rightarrow \vec{u}_1$ has a_2 in the $(j_2 =) i_2^{\text{th}}$ coordinate

a_3 in the $(j_3 =) i_3^{\text{th}}$ coordinate, etc.

$\Rightarrow \quad 0 = a_2 = a_3 = \cdots = a_m$

So $\vec{u}_1 = \vec{v}_1$.

(E) Strike out the first row of U & V, go back to (D), get $\vec{u}_2 = \vec{v}_2$, and so on. $\square$