Lecture 30: The dot product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Define the dot product by

$$\vec{u} \cdot \vec{v} := \vec{u}^T \vec{v} = (u_1 \cdots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \cdots + u_n v_n .$$

Properties of "$\cdot$":

(a) [commutativity] $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) [bilinearity] $\vec{u} \cdot (a\vec{v} + b\vec{w}) = a\,\vec{u} \cdot \vec{v} + b\,\vec{u} \cdot \vec{w}$

(c) [positive definite] $\vec{u} \cdot \vec{u} \geq 0$, with equality $\Leftrightarrow$ $\vec{u} = \vec{0}$

Ex 1/ $\vec{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix}$

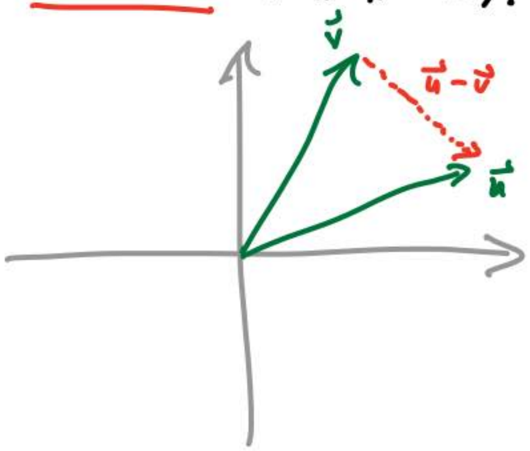
$\vec{u} \cdot \vec{u} = 30$, $\vec{v} \cdot \vec{v} = 101$, $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$ //

Next, we use (c) to define a notion of length (or norm):

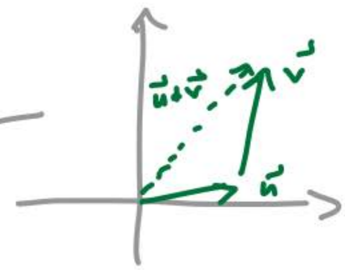
$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}$ ($\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$).

This gives a notion of distance as well:

$\text{dist}(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|$.



Properties of $\| \ \|$:

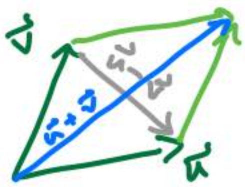


(a) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

(b) [triangle inequality] $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

(c) [Cauchy-Schwarz inequality] $|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$

(d) {parallelogram law} $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$



Let's check these:

(a) $\|c\vec{v}\|^2 = c\vec{v}\cdot c\vec{v} = c^2\,\vec{v}\cdot\vec{v} = c^2\|\vec{v}\|^2$. Take $\sqrt{\ }$.

(d) $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$$= 2\vec{u}\cdot\vec{u} + 2\vec{v}\cdot\vec{v} + \cancel{2\vec{u}\cdot\vec{v} - 2\vec{u}\cdot\vec{v}}$$

$$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

(c) $0 \le \left\| \|\vec{v}\|\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\vec{v} \right\|^2 = \left(\|\vec{v}\|\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\vec{v}\right)\cdot\left(\|\vec{v}\|\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\vec{v}\right)$

↑ we can assume $\vec{v} \ne 0$

$$= \|\vec{v}\|^2\vec{u}\cdot\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{v}\|}}\cancel{\|\vec{v}\|}\,\vec{v}\cdot\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{v}\|}}\cancel{\|\vec{v}\|}\,\vec{u}\cdot\vec{v} + \frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|^2}\vec{v}\cdot\vec{v}$$

$$= \|\vec{v}\|^2\|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v})^2 + (\vec{u}\cdot\vec{v})^2$$

$$= \|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{u}\cdot\vec{v})^2.$$

So $(\vec{u}\cdot\vec{v})^2 \le \|\vec{u}\|^2\|\vec{v}\|^2$. Take $\sqrt{\ }$'s.

(b) $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v} = \|u\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$

$$\overset{(c)}{\le} \|u\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\| = (\|\vec{u}\| + \|\vec{v}\|)^2.$$ Take $\sqrt{\ }$'s.

Ex 2/ $\vec{u}$ & $\vec{v}$ as in Example 1.

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{30}$, $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{101}$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \left\| \begin{pmatrix}2\\-5\\-1\end{pmatrix} - \begin{pmatrix}-7\\-4\\6\end{pmatrix} \right\| = \left\| \begin{pmatrix}9\\-1\\-7\end{pmatrix}\right\| = \sqrt{131}$$

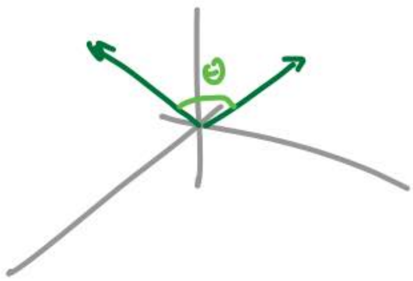
Unit vectors in the directions of $\vec{u}$ & $\vec{v}$:

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{30}}\begin{pmatrix}2\\-5\\-1\end{pmatrix}, \quad \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{101}}\begin{pmatrix}-7\\-4\\6\end{pmatrix} \quad \left(\text{works since } \frac{\vec{u}}{\|\vec{u}\|}\cdot\frac{\vec{u}}{\|\vec{u}\|} = \frac{\|\vec{u}\|^2}{\|\vec{u}\|^2} = 1\right)$$

//

We also can define the angle between 2 vectors by:

$$\theta := \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) \quad \left(\text{i.e. } \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta\right).$$



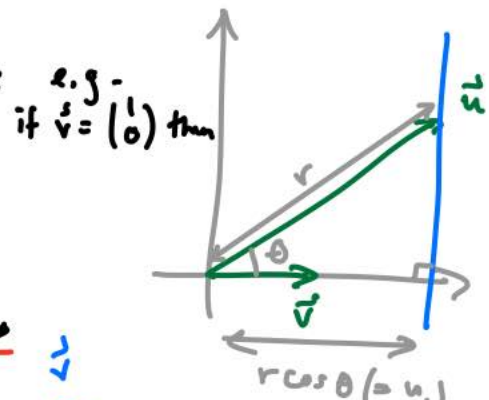
Here $\vec{u}$ & $\vec{v}$ must both be nonzero, and θ belongs to $[0,\pi]$.

This makes sense b/c Cauchy–Schwarz $\Longrightarrow \dfrac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} \le 1$

$\Longrightarrow \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \in [-1,1] \Longrightarrow \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$ is the cosine of some angle.

It is also consistent with angle in $\mathbb{R}^2$: e.g. if $\vec{v} = \begin{pmatrix}1\\0\end{pmatrix}$ then

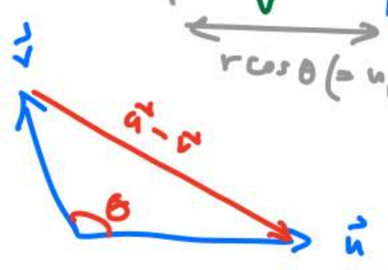
$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{\begin{pmatrix}u_1\\u_2\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{\sqrt{u_1^2+u_2^2}} = \frac{u_1}{r} = \cos\theta.$$



We have more generally the law of cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

which follows at once from the definition (try it!).



Ex 3/ $\vec{u}, \vec{v}$ as above. Then $\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

Next try $\vec{u} = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $\vec{v} = \begin{pmatrix}1\\1\\0\end{pmatrix}$. This gives

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}.$$ //

Finally we come to orthogonality:

- $\vec{u} \perp \vec{v} \overset{\text{def.}}{\iff} \vec{u}\cdot\vec{v} = 0$.

  ($\vec{u}$ is $\left\{\begin{array}{l}\text{perpendicular}\\ \text{orthogonal}\end{array}\right.$ to $\vec{v}$)

- If $W \subseteq \mathbb{R}^n$ is a subspace, the orthogonal complement of W is the subspace

$$W^\perp := \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \perp \vec{w} \text{ for every } \vec{w} \in W\}.$$

Properties: (a) [Pythagorean Theorem]

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \iff \vec{u} \perp \vec{v}$$

(b) If A is an $m \times n$ matrix

(i) $(\text{Row } A)^\perp = \text{Nul } A$

(ii) $(\text{Col } A)^\perp = \text{Nul } A^T$

(c) For $W \subset \mathbb{R}^n$, $\dim W + \dim W^\perp = n$.

(a) is clear, from the law of cosines above.

(c) I'll address in a later lecture

Turning to (b) ... write $A = \begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_m \rightarrow \end{pmatrix}$:

(i) A vector $\vec{v}$ is in $\mathrm{Nul}(A)$

$\iff \vec{r}_j \cdot \vec{v} = 0$ for each j (i.e. $\vec{v} \perp \vec{r}_j$)

$\iff \vec{v} \in \mathrm{Row}(A)^{\perp}$

(ii) $(\mathrm{Col}\, A)^{\perp} = (\mathrm{Row}\, A^T)^{\perp} \underset{(i)}{=} \mathrm{Nul}\, A^T$.

In a picture:

$$\vec{v} \longmapsto A\vec{v}$$

$$\mathbb{R}^m \longrightarrow \mathbb{R}^n$$