Lecture 35 : Abstract Inner Products

An *inner product* on a (real) vector space V is a function (or "pairing")

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$$

sending pairs of vectors $(\vec{u}, \vec{v}) \longmapsto \langle \vec{u}, \vec{v} \rangle$ to numbers

which is

- *symmetric* : $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- *bilinear* : $\langle \vec{u}, a\vec{v} + b\vec{w} \rangle = a \langle \vec{u}, \vec{v} \rangle + b \langle \vec{u}, \vec{w} \rangle$ and "vice versa"

and

- *positive definite* : $\langle \vec{u}, \vec{u} \rangle \geq 0$ with equality $\Leftrightarrow$ $\vec{u} = \vec{0}$.

This formalizes the properties of the dot product on $\mathbb{R}^n$, which is an obvious example of an inner product. But the formalization allows us to discuss orthonormal bases of spaces of functions, which is the context in which Fourier series and various "eigenstates" in quantum physics arise.

Ex 1 / $\langle \vec{u}, \vec{v} \rangle := a u_1 v_1 + b u_2 v_2 + c u_3 v_3 = \vec{u}^T \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \vec{v}$ yields an inner product on $\mathbb{R}^3$ $\Leftrightarrow$ $a, b, c > 0$, since we need to ensure that $a u_1^2 + b u_2^2 + c u_3^2 = 0 \Rightarrow u_1, u_2, u_3 = 0$. //

Ex 2/ More generally, we might ask when $\langle \vec{u}, \vec{v} \rangle := \vec{u}^{\,T} A \vec{v}$ defines an inner product on $\mathbb{R}^n$ ($A = n \times n$ matrix).

- Symmetry: $\langle \vec{v}, \vec{u} \rangle = \vec{v}^{\,T} A \vec{u} \underset{\text{it's a number!}}{=} (\vec{v}^{\,T} A \vec{u})^T = \vec{u}^{\,T} A^T \vec{v}$ equals $\langle \vec{u}, \vec{v} \rangle$ for every $\vec{u}$ & $\vec{v}$ exactly when $A = A^T$ (i.e. A is symmetric)
- bilinearity: this is easy to check (it's the distributive property of matrix mult.)
- positive definiteness: will turn out to be equivalent to all eigenvectors of A being positive (in a subsequent lecture). //

As for the dot product, define

- length by $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$
- distance by $\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$
- orthogonality by $\vec{u} \perp \vec{v} \iff \langle \vec{u}, \vec{v} \rangle = 0$.

Ex 3/ Using the dot product on $\mathbb{R}^3$ from Ex. 1 with $a = 1$, $b = 2$, $c = 3$ gives $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ since

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1 \cdot 1 \cdot 2 + 2 \cdot (-1) \cdot 1 + 3 \cdot 0 \cdot 0 = 2 - 2 = 0. \quad //$$

Properties

(1) $\|c\vec{v}\| = |c| \, \|\vec{v}\|$

(2) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (Δ inequality)

(3) $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$ (Cauchy-Schwarz inequality)

(4) $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \iff \vec{u} \perp \vec{v}$ (Pythagorean theorem)

These are proved as for the dot product: e.g., the proof of (2) uses (3), and (1) is easy. Here's (4):

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle = \langle \vec{u},\vec{u}\rangle + \langle \vec{v},\vec{v}\rangle + \langle \vec{u},\vec{v}\rangle + \langle \vec{v},\vec{u}\rangle$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u},\vec{v}\rangle, \text{ which } = \|\vec{u}\|^2 + \|\vec{v}\|^2 \Leftrightarrow \vec{u}\perp\vec{v}.$$

Gram-Schmidt

Say $W \subset V$ is a subspace with basis $\{\vec{w}_1, \dots, \vec{w}_k\}$. Then

$$\vec{v}_1 := \vec{w}_1$$
$$\vec{v}_2 := \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1$$
$$\vec{v}_3 := \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{\langle \vec{v}_2, \vec{v}_2\rangle}\vec{v}_2 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1$$
$$\vdots$$

produces an orthogonal basis of W.

Orthogonal Projection

If $\{\vec{w}_1, \dots, \vec{w}_k\}$ is an orthogonal basis of W, then

$$\operatorname{proj}_W \vec{y} := \frac{\langle \vec{w}_1, \vec{y}\rangle}{\langle \vec{w}_1, \vec{w}_1\rangle}\vec{w}_1 + \cdots + \frac{\langle \vec{w}_k, \vec{y}\rangle}{\langle \vec{w}_k, \vec{w}_k\rangle}\vec{w}_k \quad \text{satisfies}$$

(i) $(\vec{y} - \operatorname{proj}_W \vec{y}) \perp W$ and $\operatorname{proj}_W \vec{y} \in W$ (check $\langle \vec{y} - \operatorname{proj}_W \vec{y}, \vec{w}_i\rangle$ is 0 for $i = 1, \dots, k$)

(ii) $\operatorname{proj}_W \vec{y}$ gives the closest vector (to $\vec{y}$) in W

$$\Big[\, \operatorname{dist}(\vec{y},\vec{w})^2 = \|\underbrace{(\vec{y} - \operatorname{proj}_W \vec{y})}_{\text{in } W^\perp} + \underbrace{(\operatorname{proj}_W \vec{y} - \vec{w})}_{\text{in } W}\|^2 \underset{\text{Pythagorean Thm}}{\overset{(4)}{=}} \|\vec{y} - \operatorname{proj}\vec{y}\|^2 + \|\operatorname{proj}\vec{y} - \vec{w}\|^2$$
$$\geq \|\vec{y} - \operatorname{proj}_W \vec{y}\|^2 = \operatorname{dist}(\vec{y}, \operatorname{proj}_W \vec{y})^2 \,\Big]$$

(iii) $\|\vec{y}\| \geq \|\operatorname{proj}_W \vec{y}\|$ (projection decreases length), with equality $\Leftrightarrow \vec{y} \in W$.

$$\Big[\, \|\vec{y}\|^2 \underset{(4)}{=} \|\overbrace{\vec{y} - \operatorname{proj}\vec{y}}^{\text{in } W^\perp}\|^2 + \|\overbrace{\operatorname{proj}\vec{y}}^{\text{in } W}\|^2 \geq \|\operatorname{proj}_W \vec{y}\|^2, \text{ again using Pythagoras} \,\Big]$$

Ex 4/ On $V = \mathbb{P}_2$, define $\langle \overset{\text{polynomials}}{p, q} \rangle := p(t_1)q(t_1) + p(t_2)q(t_2) + p(t_3)q(t_3)$

where t_1, t_2, t_3 are any 3 distinct points (which we have fixed).

This is symmetric & bilinear. To see positive-definite, note that

$$\langle p,p\rangle = p(t_1)^2 + p(t_2)^2 + p(t_3)^2 \geq 0 \quad \text{with equality} \iff p \text{ is } 0 \text{ at } t_1, t_2, t_3$$

(a nonzero quadratic polynomial can't have 3 roots) $\iff$ p is identically 0.

So suppose $t_1 = 0$, $t_2 = 1$, & $t_3 = 2$. How can we compute the orthogonal projection of $Q := t^2$ on $W := \text{span}\{\underset{p_1}{1}, \underset{p_2}{t}\}$?

<u>Step 1</u>: Gram-Schmidt: $q_1 = p_1 = 1$

$$q_2 = p_2 - \frac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1 = t - \frac{\overbrace{\langle t, 1\rangle}^{0\cdot 1 + 1\cdot 1 + 2\cdot 1 = 3}}{\underbrace{\langle 1, 1\rangle}_{1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3}} 1 = t - 1.$$

<u>Step 2</u>: $$\text{proj}_W Q = \frac{\overbrace{\langle Q, q_1\rangle}^{0^2\cdot 1 + 1^2\cdot 1 + 2^2\cdot 1 = 5}}{\underbrace{\langle q_1, q_1\rangle}_{3}} q_1 + \frac{\overbrace{\langle Q, q_2\rangle}^{0^2\cdot(0-1) + 1^2(1-1) + 2^2(2-1) = 4}}{\underbrace{\langle q_2, q_2\rangle}_{(0-1)^2 + (1-1)^2 + (2-1)^2 = 2}} q_2 = \frac{5}{3} q_1 + \frac{4}{2} q_2$$

$$= \frac{5}{3}\cdot 1 + 2\cdot(t-1) = 2t - \frac{1}{3}.$$

<u>Proof of Cauchy-Schwarz</u>: If $\vec{u} = \vec{0}$, easy. Otherwise, let $W = \text{span}\{\vec{u}\}$. Then $\|\vec{v}\| \geq \|\text{proj}_W \vec{v}\| = \left\| \frac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle} \vec{u} \right\| \underset{(1)}{=} \frac{|\langle \vec{v}, \vec{u}\rangle|}{\langle \vec{u}, \vec{u}\rangle} \|\vec{u}\|$

$= \frac{|\langle \vec{v}, \vec{u}\rangle|}{\|\vec{u}\|^{\cancel{2}}} \cancel{\|\vec{u}\|} = \frac{|\langle \vec{v}, \vec{u}\rangle|}{\|\vec{u}\|}$. So $\|\vec{v}\|\|\vec{u}\| \geq |\langle \vec{v}, \vec{u}\rangle|$. $\square$

Ex 5/ $\overset{\infty\text{-dimensional vector space}}{V} = C[-1, 1]$ = continuous functions on the interval $[-1, 1] \subset \mathbb{R}$

$\langle f, g\rangle := \int_{-1}^{1} f(t) g(t)\, dt$ is symmetric & bilinear.

Why positive-definite? $\langle f, f\rangle = \int_{-1}^{1} (f(t))^2 dt \geq 0$ since $f^2 \geq 0$,

and by a theorem from Calculus, if the integral $= 0$ then $f(t) \equiv 0$.

Consider the finite-dimensional subspace $W \subset V$ spanned by $\underset{p_1}{1}, \underset{p_2}{t}, \underset{p_3}{t^2}$.

What is an orthonormal basis? Start w/ <u>orthogonal</u>:

$$q_1 = p_1 = 1$$

$$q_2 = p_2 - \frac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1 = t - \frac{\int_{-1}^{1} t\cdot 1\, dt}{\int_{-1}^{1} 1\cdot 1\, dt}\cdot 1 = t - \frac{0}{2} = t$$

$$q_3 = p_3 - \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2 - \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 = t^2 - \frac{\cancelto{0}{\int_{-1}^{1} t^2 \cdot t\, dt}}{\int_{-1}^{1} t\cdot t\, dt}\, t - \frac{\overbrace{\int_{-1}^{1} t^2\cdot 1\, dt}^{2/3}}{\underbrace{\int_{-1}^{1} 1\cdot 1\, dt}_{2}}\, 1$$

$$= t^2 - \frac{1}{3}.$$

Now

$$\|q_1\|^2 = \int_{-1}^{1} 1\cdot 1\, dt = 2, \quad \|q_2\|^2 = \int_{-1}^{1} t\cdot t\, dt = \frac{2}{3}, \quad \|q_3\|^2 = \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt = \frac{8}{45}$$

$$\Rightarrow u_1 = \frac{q_1}{\|q_1\|} = \frac{1}{\sqrt{2}}, \quad u_2 = \frac{q_2}{\|q_2\|} = \sqrt{\frac{3}{2}}\, t, \quad u_3 = \frac{q_3}{\|q_3\|} = \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right).$$