Lecture 37: Quadratic Forms

Definition: A quadratic form on $\mathbb{R}^n$ is a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form $f(\vec{x}) = \vec{x}^T A \vec{x}$ for some symmetric (real, $n\times n$) matrix A.

called the matrix of the form f

Ex 1/ Compute $\vec{x}^T A \vec{x}$, for $A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

$$(x_1\ x_2\ x_3)\begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1\ x_2\ x_3)\begin{pmatrix} 4x_1+3x_2 \\ 3x_1+2x_2+x_3 \\ x_2+x_3 \end{pmatrix} = 4x_1^2+2x_2^2+x_3^2 + 3x_1x_2+3x_2x_1 + x_2x_3+x_3x_2$$

$3x_1x_2+3x_2x_1 = 6x_1x_2$, $\quad x_2x_3+x_3x_2 = 2x_2x_3$

In general,

$$(x_1 \cdots x_n)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1 \cdots x_n)\begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix} = \sum_{i,j=1}^n x_i a_{ij} x_j$$

$$= \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i<j} a_{ij}x_ix_j + \sum_{i>j} a_{ij}x_ix_j$$

$$\sum_{i>j} a_{ij}x_ix_j = \underset{\substack{\left(\text{swap}\atop i\,\&\,j\right)}}{=} \sum_{i<j} a_{ji}x_jx_i \underset{(A=A^T)}{=} \sum_{i<j} a_{ij}x_ix_j$$

$$= \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i<j}(2a_{ij})x_ix_j.$$

Ex 2/ Find the matrix of each form:

- (on $\mathbb{R}^2$) $x_1^2+x_2^2 = (x_1\ x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T I_2 \vec{x} \Rightarrow A = I_2$
- x_1x_2 $\left[= (x_1\ x_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ ain't symmetric.}\right]$
  $= \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1 = (x_1\ x_2)\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$
- $10x_1^2 - 6x_1x_2 - 3x_2^2 = (x_1\ x_2)\begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}$

(on $\mathbb{R}^3$) • $8x_1^2 + 7x_2^2 - 3x_3^2 \underbrace{- 6x_1x_2 + 4x_1x_3 - 2x_2x_3}_{\substack{\text{think: } -3x_1x_2 - 3x_2x_1 + 2x_1x_3 + 2x_3x_1 \\ -x_2x_3 - x_3x_2}} \Rightarrow A = \begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}$

Level sets of quadratic forms

What does the solution set of

$$8x_1^2 - 4x_1x_2 + 5x_2^2 = 1$$

look like? It's an ellipse, but for that to become clear you'll need to perform a rotation of coordinates. Begin by recognizing the left-hand side as a quadratic form

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T A \vec{x} = Q(\vec{x}),$$

where A is symmetric.

But let's step back for a moment and look more generally at the equation

(*) $$\vec{x}^T A \vec{x} = 1 \leftarrow \text{or any positive number}$$

with A symmetric $n \times n$. By the Spectral Theorem (Lecture 36), there is an orthonormal eigenbasis $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ for A, so that $P_{\mathcal{B}}^{-1} = P_{\mathcal{B}}^T$ and

$$A = P_{\mathcal{B}} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{D} P_{\mathcal{B}}^T$$

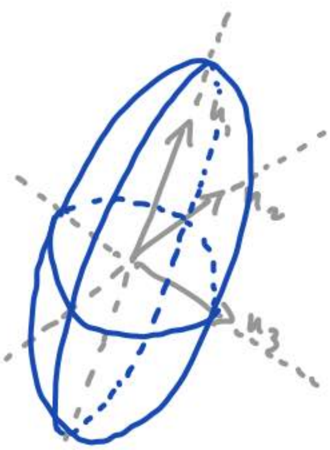
Writing $\vec{y} = [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = P_{\mathcal{B}}^T \vec{x}$ for the "eigencoordinates", we have

$$\vec{x}^T A \vec{x} = \vec{x}^T P_{\mathcal{B}} D P_{\mathcal{B}}^T \vec{x} = \left(P_{\mathcal{B}}^T \vec{x}\right)^T D \left(P_{\mathcal{B}}^T \vec{x}\right) = \vec{y}^T D \vec{y},$$

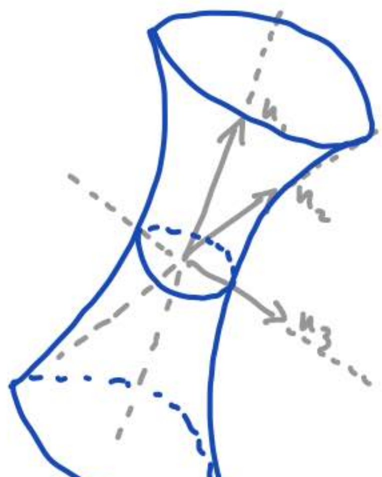
and our equation (*) becomes

$$(**) \qquad \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1.$$

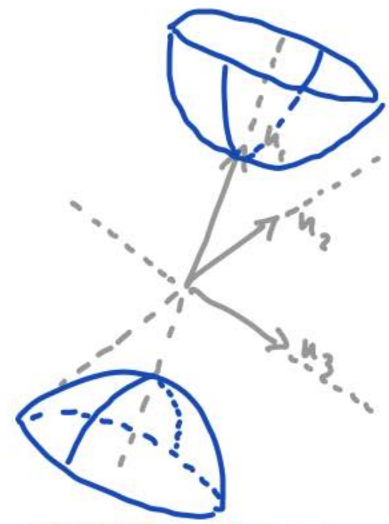
To interpret this geometrically for $n=3$, assume that $\lambda_1 \le \lambda_2 \le \lambda_3$. If all 3 eigenvalues are positive, then (**) will define an ellipsoid with principal axes (in the directions of $\vec{u}_1, \vec{u}_2, \vec{u}_3$) of lengths $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}$. If only $\lambda_2, \lambda_3 > 0$, then (**) is an elliptic hyperboloid; and if only $\lambda_3 > 0$, (**) is a hyperboloid of 2 sheets.



ELLIPSOID



ELLIPTIC HYPERBOLOID



HYPERBOLOID OF TWO SHEETS

Back to $n=2$, and our original equation: A orthogonally diagonalizes

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

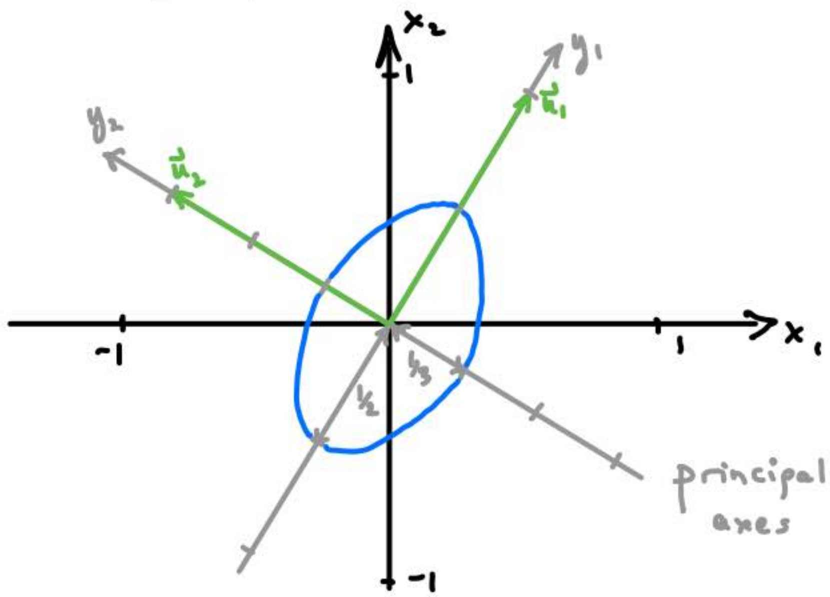
and so if (y_1, y_2) are coordinates along the axes defined by the (unit) eigenvectors

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

our equation becomes

$$1 = 4y_1^2 + 9y_2^2 = \frac{y_1^2}{(1/2)^2} + \frac{y_2^2}{(1/3)^2}.$$

Now it is really easy to sketch the solution:



<u>Q</u> : What sorts of figures do you get if

$(n=2)$ $\lambda_2 > 0$ and $\lambda_1 = 0$

$(n=3)$ $\lambda_3 > 0$, $\lambda_2 = 0$, $\lambda_1 < 0$?

Quadratic forms as functions

Suppose $A = PDP^T$ is an orthogonal diagonalization. Then

(†) $Q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T P D P^T \vec{x} = (P^T \vec{x}) D (P^T \vec{x})$
$= \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$

as before. So it's pretty clear that Q is "controlled by the eigenvalues λ_i" once you do the orthogonal change of coordinates.

<u>Definition</u>: (i) Q is positive definite $\iff$ $Q(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0}$

positive semidefinite $\iff$ $Q(\vec{x}) \geq 0 \quad \forall \vec{x}$

indefinite $\iff$ Q assumes both positive & negative values on nonzero $\vec{x}$'s

negative semidefinite $\iff$ $Q(\vec{x}) \leq 0 \quad \forall \vec{x}$

negative definite $\iff$ $Q(\vec{x}) < 0 \quad \forall \vec{x} \neq \vec{0}$

(ii) A is positive definite $\iff$ all $\lambda_i > 0$

positive semidefinite $\iff$ all $\lambda_i \geq 0$

indefinite $\iff$ A has both positive & negative eigenvalues

negative semidefinite $\iff$ all $\lambda_i \leq 0$

negative definite $\iff$ all $\lambda_i < 0$.

<u>Theorem</u>: Q is $\boxed{}$ $\iff$ A is $\boxed{}$

(Insert same term in both places.)

<u>Proof</u>: Clear from (†). e.g., if λ_i's are > 0, then
$Q(\vec{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \geq 0$, and $= 0 \iff \vec{y} = \vec{0} \iff \vec{x} = \vec{0}$. $\square$

Quadratic forms and inner products

Let A be an $n \times n$ symmetric matrix.

Corollary (to the Theorem): $\langle \vec{x}, \vec{y} \rangle := \vec{x}^T A \vec{y}$ is an inner product on $\mathbb{R}^n$ $\iff$ A is positive definite.

Proof: A is positive-definite $\overset{\text{Thm.}}{\iff}$ $Q(\vec{x}) := \langle \vec{x}, \vec{x} \rangle$ is positive-definite
(eigenvalues > 0) ($\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality $\iff \vec{x} = \vec{0}$)

$\iff$ the symm. bilinear form $\langle \cdot , \cdot \rangle$ satisfies the positive-definiteness property

$\iff$ $\langle \cdot , \cdot \rangle$ is an inner product. $\square$

More geometrically,

$\vec{x}^T A \vec{y}$ gives an inner product

$\iff$

the set $\|\vec{x}\| = 1$ (comprising elements of "length" 1) is an ellipsoid.