Lecture 5: Parametrizing solutions

Review of "Span"

Recall that the span of a collection of vectors is the set of all linear combinations of them:

Ex 1/

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\\-2\\-1\end{pmatrix}, \begin{pmatrix}-1\\3\\-1\\0\end{pmatrix}\right\} = \left\{ s\begin{pmatrix}1\\0\\-2\\-1\end{pmatrix} + t\begin{pmatrix}-1\\3\\-1\\0\end{pmatrix} \;\middle|\; s,t \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}s\\0\\-2s\\-s\end{pmatrix} + \begin{pmatrix}-t\\3t\\-t\\0\end{pmatrix} \;\middle|\; s,t\in\mathbb{R}\right\} = \left\{\begin{pmatrix}s-t\\3t\\-2s+3t\\-s\end{pmatrix} \;\middle|\; s,t\in\mathbb{R}\right\} //$$

So if you are asked

"Taking W = set of all vectors of the form $\begin{pmatrix}3s-4t\\-4s+4t\\-4s-4t\\-3s+5t\end{pmatrix}$, find $\vec{u}$ & $\vec{v}$ such that $W = \operatorname{span}\{\vec{u},\vec{v}\}$"

you'll just reverse the process, obtaining

$$\vec{u} = \begin{pmatrix}3\\-4\\-4\\-3\end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix}-4\\4\\-4\\5\end{pmatrix}.$$

Homogeneous linear systems

These are systems of the form

$$A\vec{x} = \vec{0}.$$

They always have the "trivial solution" $\vec{x} = \vec{0}$, and the solution set is always the span of a set of vectors (as we now demonstrate).

Ex 2/ Determine all solutions of $\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ \qquad -3x_2 - 3x_3 = 0. \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

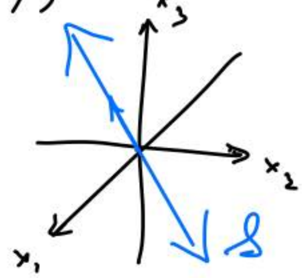
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{rref}$$

x_3 free

[N.B.: You may drop the right-hand column of 0's if you are comfortable doing that.]

So $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases} \implies \vec{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$,

and the solution set is $S = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.



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Inhomogeneous linear systems

These are the systems $A\vec{x} = \vec{b}$ with $\vec{b} \neq \vec{0}$.

In this case, $\vec{x} = \vec{0}$ is NEVER a solution. So the solution set $\mathcal{S}$ can't be a span: if nonempty, it will be a "parallel translate" of a span.

Ex 3/ Determine all solutions of
$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -4x_1 - 4x_2 - 8x_3 = -16 \\ \quad\quad -3x_2 - 3x_3 = 12. \end{cases}$$

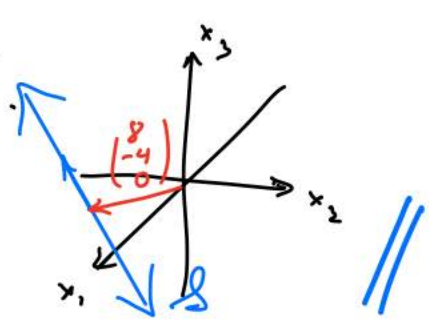
$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{rref}$$

x_3 free

$$\Longrightarrow \begin{cases} x_1 = 8 - x_3 \\ x_2 = -4 - x_3 \end{cases} \Longrightarrow \vec{x} = \begin{pmatrix} 8 - x_3 \\ -4 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \mathcal{S} = \left\{ \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$



Here is another homogeneous system problem, with the matrix already nearly row-reduced:

Ex 4/ If $A \underset{\text{row-equiv.}}{\sim} \begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, describe the solution set to $A\vec{x} = \vec{0}$.

$$\longrightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 3 & -7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

$$\begin{array}{cccccc} & \uparrow & & \uparrow & & \uparrow \\ & x_2 & & x_4 & & x_6 \text{ free} \end{array}$$

Now mentally adjoining a column of zeros to the right, we get

$$\begin{cases} x_1 = 4x_2 - 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{cases} \Longrightarrow$$

$$S = \left\{ \begin{pmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{pmatrix} \middle| \; x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \middle| \; x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}. \quad //$$

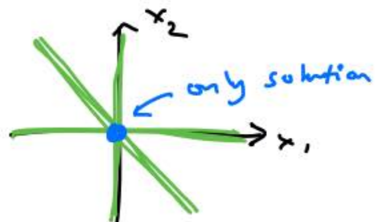
Relation between homogeneous & inhomogeneous systems

Consider $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}\ (\neq \vec{0})$, with solution sets

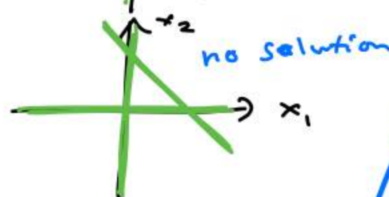
$$\mathcal{S}_{\vec{0}} \quad \text{and} \quad \mathcal{S}_{\vec{b}}.$$

Case I: $\mathcal{S}_{\vec{b}}$ empty

Ex 5/
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longleftrightarrow$$



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longleftrightarrow$$



Case II: $\mathcal{S}_{\vec{b}}$ nonempty. Let $\vec{p} \in \mathcal{S}_{\vec{b}}$ be <u>any</u> solution (of the inhomogeneous system).

Then $\boxed{\mathcal{S}_{\vec{b}} = \mathcal{S}_{\vec{0}} + \vec{p}} = \{ \vec{w} + \vec{p} \mid \vec{w} \in \mathcal{S}_{\vec{0}} \}$ is the translate of the homogeneous solution set by $\vec{p}$.

Why?

- If $\vec{w} \in \mathcal{S}_{\vec{0}}$, then $A(\vec{w}+\vec{p}) = A\vec{w} + A\vec{p} = \vec{0} + \vec{b} = \vec{b}$. So $\mathcal{S}_{\vec{0}} + \vec{p} \subseteq \mathcal{S}_{\vec{b}}$.
- If $\vec{x} \in \mathcal{S}_{\vec{b}}$, then write $\vec{x} = (\vec{x} - \vec{p}) + \vec{p}$. Since $A(\vec{x}-\vec{p}) = A\vec{x} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$, we see that $\vec{x} - \vec{p} \in \mathcal{S}_{\vec{0}}$. So $\vec{x} \in \mathcal{S}_{\vec{0}} + \vec{p}$. This shows the reverse inclusion $\mathcal{S}_{\vec{b}} \subseteq \mathcal{S}_{\vec{0}} + \vec{p}$.