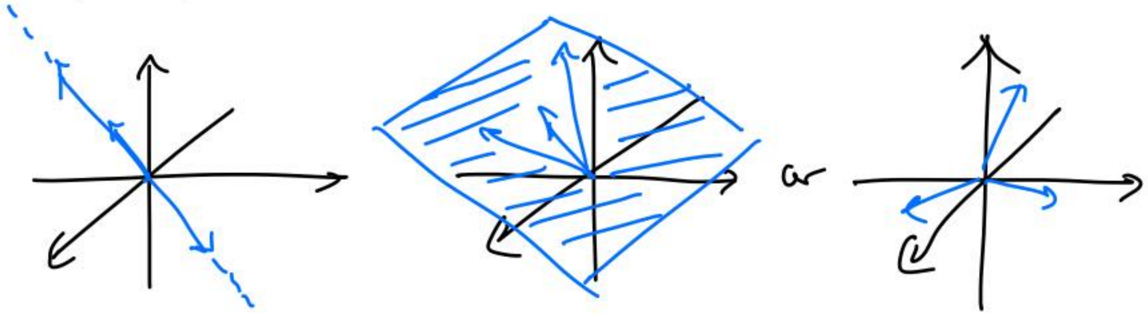
Lecture 6: Linear Independence

Ex 1/ Consider the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$ in $\mathbb{R}^3$. Do they span a line, a plane, or all of space? That is, do they look like



To find out, we could ask "for what kinds of $\vec{b}$ is

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + z \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \vec{b}$$

consistent?"

Writing this as an augmented matrix and row-reducing gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & -\frac{1}{3}b_2 + \frac{4}{3}b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{5}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & 2 & -\frac{1}{3}b_2 + \frac{4}{3}b_1 \\ 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right]$$

So our vectors span the <u>plane described by $b_1 - 2b_2 + b_3 = 0$.</u> We can also write this as the span of $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (why?). //

Definition: Let $\vec{v}_1, \dots, \vec{v}_n$ be a finite sequence (or "indexed set") of vectors in $\mathbb{R}^m$. A linear dependence relation (or "linear dependency") among these vectors is an equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

in which $c_1, \dots, c_n$ are real numbers that are not all zero. A sequence of vectors with such a relation is called linearly dependent.

Ex 1 (cont'd) / Let's show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent: we must solve the equation $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Row-reducing the augmented matrix yields

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x = z \\ y = -2z. \end{array}$$

Picking any nonzero z will do — say, $z = 1$. Then we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \rightsquigarrow 1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad //$$

Ex 2/ The vectors $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ have span all of $\mathbb{R}^3$, and clearly also have no linear dependence relation. How should we phrase the latter property?

//

Definition: A sequence of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^m$ is **linearly independent** if it is not linearly dependent: that is, if the only solution to

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{0}$$

is $\vec{x} = \vec{0}$. So: to check that $\vec{v}_1, \dots, \vec{v}_n$ are independent, you must show the implication

$$\left.\begin{matrix} c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \vec{0} \\ (c_1, \dots, c_n \in \mathbb{R}) \end{matrix}\right\} \Longrightarrow c_1 = \cdots = c_n = 0.$$

Ex 2 (cont'd)/ Suppose $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$. Therefore $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent.

//

<u>Two equivalent conditions</u> on columns $\vec{v}_1, \dots, \vec{v}_n$ of an $m \times n$ matrix A:

(I) The columns of A are linearly independent.

(II) All columns of $\mathrm{rref}(A)$ contain a leading '$\underline{1}$'.
(i.e., all columns of A are pivot columns)

Check: $\underline{(\text{II}) \Rightarrow (\text{I})}$: Suppose $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$.

Since (II) holds, row-reducing $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$

yields $\left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \end{array}\right]$ (might be zeroes down here) $\Longrightarrow$ only solution is $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{cases}$.

$\underline{\neg(\text{II}) \Rightarrow \neg(\text{I})}$: If $\mathrm{rref}\left[\begin{array}{c|c} A & \vec{0} \end{array}\right] = \begin{array}{c} \left[\begin{array}{cccccc|c} 1 & \ddots & & * & & & 0 \\ & & 1 & * & & & 0 \\ & & & & 1 & & 0 \\ & & & & & \ddots\; 1 & \vdots \\ & & & & & & 0 \end{array}\right] \\ \quad\quad\quad\quad x_i \end{array}$

has a non-pivot column (say, the i^{th}),
then there exists a solution to $\underbrace{x_1\vec{v}_1 + \dots + x_n\vec{v}_n}_{\text{i.e. } A\vec{x}} = \vec{0}$

in which x_i can be anything we want (in particular, nonzero). This gives a linear dependence relation.

$\square$

Ex 3/ Are the vectors $\begin{pmatrix}1\\1\\2\\4\end{pmatrix}, \begin{pmatrix}2\\-1\\-5\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\-4\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\\6\end{pmatrix}$ linearly independent in $\mathbb{R}^4$?

Row-reduce:

$$\begin{pmatrix}1 & 2 & 1 & 2\\ 1 & -1 & -1 & 1\\ 2 & -5 & -4 & 1\\ 4 & 2 & 0 & 6\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 1 & 2\\ 0 & -3 & -2 & -1\\ 0 & -9 & -6 & -3\\ 0 & -6 & -4 & -2\end{pmatrix} \longrightarrow \begin{pmatrix}\boxed{1} & 0 & -\frac{1}{3} & \frac{4}{3}\\ 0 & \boxed{1} & \frac{2}{3} & \frac{1}{3}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \rightsquigarrow \underline{\text{NO}}!$$

[To find a linear dependency, take for example $x_3 = 3$, $x_4 = 0$, so $x_1 = 1$ and $x_2 = -2$: $1\begin{pmatrix}1\\1\\2\\4\end{pmatrix} - 2\begin{pmatrix}2\\-1\\-5\\2\end{pmatrix} + 3\begin{pmatrix}1\\-1\\-4\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.] //

<u>Notice that</u> (as above, $A = m \times n$ matrix):

(1) If $m < n$ (more than m vectors in $\mathbb{R}^m$), then the columns of A are <u>never</u> independent.

(Because there are only m rows to have leading '1''s, and there are more columns than that.)

(2) If $m = n$ (square matrix), then

columns are independent $\iff$ $\text{rref}(A) = \begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$

$\iff$ columns span $\mathbb{R}^m$.

(The last equivalence follows from Lecture 4, since the $m \times m$ identity matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ has no rows of "all 0" at the bottom.)

(3) If a sequence of vectors $\vec{v}_1, \ldots, \vec{v}_n$ contains $\vec{0}$, then that sequence is dependent.

(Say $\vec{v}_1 = \vec{0}$. Then $1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \ldots + 0\,\vec{v}_n = \vec{0}$ is a linear dependency.)

(4) A sequence is linearly dependent $\iff$ at least one of the $\vec{v}_i$ is a linear combination of the others.

(If $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$, with some $c_k \neq 0$, then by rescaling we can assume $c_k = 1$. Then

$$\vec{v}_k = -c_1\vec{v}_1 - \ldots - c_{k-1}\vec{v}_{k-1} - c_{k+1}\vec{v}_{k+1} - \ldots - c_n\vec{v}_n.$$

The converse is also clear from this.)

WARNING: Consider $\underset{\vec{v}_1}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \underset{\vec{v}_4}{\begin{pmatrix}0\\0\\1\end{pmatrix}}$.

This is linearly dependent, and $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are each linear combinations of the others. But $\vec{v}_4$ isn't. So (4) means exactly what it says ("at least one", not "every").

(5) Finally, why do I not speak simply of the <u>set</u> of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ being (in)dependent? I say "finite <u>sequence</u>"; the book "<u>indexed</u> set". This is because repetitions matter, and a plain "set" doesn't remember repetitions. If, for example, $A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$ and $\vec{v}_1 = \vec{v}_2 \, (\neq \vec{0})$, but $\vec{v}_3$ is not a multiple of $\vec{v}_1$, the <u>set</u> of column vectors is just $\{\vec{v}_2, \vec{v}_3\}$.

But (i) $\vec{v}_2, \vec{v}_3$ is independent

while

(ii) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is <u>dependent</u> b/c of the relation

$$1\vec{v}_1 + (-1)\vec{v}_2 + 0\vec{v}_3 = 0.$$

In this situation, the theorems would all be wrong if we considered columns of A to be independent: we <u>must</u> keep track of repetitions, and it is (ii) that matters.

So: to be absolutely clear — a sequence of vectors $\vec{v}_1, \dots, \vec{v}_n$ in which the same vector appears twice is ALWAYS <u>dependent</u>.