Lecture 1 : Normal families

We start off the semester with some analytic machinery related to families of holomorphic functions, with a view towards the Riemann Mapping Theorem.

I. Families of continuous functions

Let $\Omega \subset \mathbb{C}$ be a region, $\mathcal{E} \subset \Omega$ a subset, and $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ a family (i.e. set) of complex-valued functions on Ω.

Definition $\mathcal{F}$ is equicontinuous on $\mathcal{E}$ $\iff$

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad$ s.t.

$$\left.\begin{array}{l} |z - w| < \delta \\ z, w \in \mathcal{E} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} |f(z) - f(w)| < \epsilon \\ \forall f \in \mathcal{F}. \end{array}\right.$$

Definition (i) A sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{F}$ converges normally $\iff$ it converges uniformly on (all) compact subsets.

(ii) $\mathcal{F}$ is <u>normal</u> $\iff$

every sequence has a normally convergent subsequence.

$\boxed{\text{WARNING}}$ In (i), the limit function need <u>not</u> belong to $\mathcal{F}$.

$\boxed{\text{Arzela-Ascoli Lemma}}$ <u>Assume $\mathcal{F}$ consists of continuous functions. Then : $\mathcal{F}$ is normal $\iff$</u>

(a) <u>$\mathcal{F}$ is equicontinuous on compact subsets $K \subset \Omega$</u>

AND

(b) <u>$\mathcal{F}$ is pointwise bounded</u> (for every $z \in \Omega$, $\exists\, M_z$ s.t. $|f(z)| \leq M_z$ $(\forall f \in \mathcal{F})$).

<u>Proof of $(\Longrightarrow)$:</u>

(a) $\mathcal{F}$ not equicontinuous on $K \subset \Omega \Longrightarrow$

$\exists\ \epsilon > 0$ & sequences $\{z_n\}, \{w_n\} \subset K$, $\{f_n\} \subset \mathcal{F}$ s.t.

$(\forall n)$ $|z_n - w_n| < \frac{1}{n}$ and $|f_n(z_n) - f_n(w_n)| \geq \epsilon$.

Since K is compact & $\mathcal{F}$ is normal, $\exists\, \{n_k\}$ s.t.

$z_{n_k}, w_{n_k} \longrightarrow x \in K$

$f_{n_k}|_K \longrightarrow f$ uniformly, so that f is (uniformly) C^0.

Therefore, $\exists\ k_0$ s.t. $k \geq k_0 \Longrightarrow$

$$|f_{n_k}(z_{n_k}) - f_{n_k}(w_{n_k})| \leq |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(w_{n_k})| + |f(w_{n_k}) - f_{n_k}(w_{n_k})|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \quad \text{a contradiction.}$$

(b) Fix $z \in \Omega$, and let

$$\{w_n\} \subset \overline{\{f(z) \mid f \in \mathcal{F}\}} =: \mathcal{S} \text{ be a sequence}$$

and $\{f_n\} \subset \mathcal{F}$ be such that $|f_n(z) - w_n| < \frac{1}{n}$.

$\mathcal{F}$ normal $\Longrightarrow \exists\ \{f_{n_k}\}$ convergent

$\Longrightarrow \{f_{n_k}(z)\}$ convergent (limit $\in \mathcal{S}$)

$\Longrightarrow \{w_{n_k}\}$ convergent (to same limit).

Hence, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure & is $\therefore$ bounded.

Proof of ($\Longleftarrow$): Consider $\{f_n\} \subseteq \mathcal{F}$.

Let $E = \{z_1, z_2, \ldots\} \subset \Omega$ be a countable dense subset. Fix $k \geq 1$, and inductively construct

$$\mathcal{S}_k := \{n_j^{[k]}\}_{j \in \mathbb{N}}$$

by extracting a convergent subsequence from the bounded (by (b)) sequence $\{f_{n_j^{[k-1]}}(z_k)\} \subset \mathbb{C}$. We get

$$\mathbb{N} \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots .$$

Taking $\mathcal{S}_\infty := \{n_\ell^{[\ell]}\}_{\ell \in \mathbb{N}} =: \{r_\ell\}$, it has at most $(k-1)$ terms not in $\mathcal{S}_k$. Hence, $\lim_{\ell \to \infty} f_{r_\ell}(z_k)$ exists (in $\mathbb{C}$) for each k.

Now given $\epsilon > 0$ and $K \subset \Omega$ compact, there exists $\delta > 0$ s.t.
by (a)

$$|z - w| < \delta \implies |f_{r_\ell}(z) - f_{r_\ell}(w)| < \epsilon/3 \quad (\forall \ell).$$

Cover K with disks $B_1, \ldots, B_m$ of radius $\delta/2$.

$$\begin{aligned}
\text{Density of } E &\implies \exists\, p_i \in B_i \cap E \quad (\forall i) \\
&\implies \lim_{\ell \to \infty} f_{r_\ell}(p_i) \text{ exists} \quad (\forall i) \\
&\implies \exists\, N \text{ s.t.} \\
&\qquad \ell, m \geq N \implies |f_{r_\ell}(p_i) - f_{r_m}(p_i)| < \tfrac{\epsilon}{3} \ (\forall i).
\end{aligned}$$

For any $x \in K$,

$$\begin{aligned}
x \in B_i &\implies |x - p_i| < \delta \\
&\implies \text{for } \ell, m \geq N, \\
&\qquad |f_{r_\ell}(x) - f_{r_m}(x)| \leq |f_{r_\ell}(x) - f_{r_\ell}(p_i)| + |f_{r_\ell}(p_i) - f_{r_m}(p_i)| \\
&\qquad\qquad + |f_{r_m}(p_i) - f_{r_m}(x)| \\
&\qquad < \epsilon.
\end{aligned}$$

Hence f_{r_ℓ} converges uniformly on K. $\square$

II. Families of holomorphic functions

Definition $\mathcal{F}$ locally bounded $\iff$

the $\{f_\alpha\}$ are uniformly bounded on compact subsets: i.e., for each compact $K \subset \Omega$, $\exists\, M_K$ s.t.

$$|f(z)| \le M_K \quad (\forall z \in K \;\&\; f \in \mathcal{F}).$$

Montel's Lemma Assume $\mathcal{F}$ consists of holomorphic functions. Then

$$\mathcal{F} \text{ normal} \iff \mathcal{F} \text{ locally bounded.}$$

Remark: For $\mathcal{F}$ holomorphic family, normally convergent sequences have holomorphic limit functions. (Recall that this is essentially Morera's Theorem.)

Proof of ($\Longrightarrow$): Given $\epsilon > 0$ and $K \subset \Omega$ compact.

By Arzela-Ascoli, $\mathcal{F}$ is equicontinuous on K with $\delta > 0$ such that

$$|z - w| < \delta \implies |f(z) - f(w)| < \epsilon \quad (\forall f \in \mathcal{F}).$$

Cover K with a finite collection $D_i = D(w_i, \delta/2)$ of disks. Then for $z \in D_i$,

$$|f(z)| \leq M_{w_i} + \epsilon \quad (\forall f \in \mathcal{F}),$$

where M_{w_i} are the (pointwise) bounds at each w_i guaranteed by Arzela-Ascoli. Conclude that $\mathcal{F}$ is uniformly bounded by $\max_i \{M_{w_i} + \epsilon\}$ on K.

Proof of ($\Leftarrow$): Certainly, local boundedness implies pointwise boundedness. To apply Arzela-Ascoli, we must prove equicontinuity on compact subsets.

Let $K \subset \Omega$ be one, and (taking $r < d(K, \Omega^c)$) let $\{D_i = D(w_i, r/4)\}$ be a finite collection covering K, with $\mathcal{F}$ bounded by M_i on $\Delta_i = \overline{D}(w_i, r)$.

using local boundedness

Fix $\epsilon > 0$, and put $M := \max\{M_i\}$, $\delta := \min\{\frac{r}{4}, \frac{\epsilon r}{4M}\}$. Given $z, w \in K$ with $|z-w| < \delta$, $\exists\, i$ s.t. $|w - w_i| < \frac{r}{4}$, and so $|z - w_i| < \delta + \frac{r}{4} \leq \frac{r}{2}$.

Now for all $f \in \mathcal{F}$, by Cauchy

$$f(z) - f(w) = \frac{1}{2\pi i}\oint_{\partial\Delta_i}\left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w}\right) d\zeta$$
$$= \frac{z-w}{2\pi i}\oint_{\partial\Delta_i}\frac{f(\zeta)}{(\zeta-z)(\zeta-w)}\,d\zeta$$

$$\Rightarrow |f(z)-f(w)| \le \frac{|z-w|}{2\pi}\cdot\frac{2\pi r\, M_i}{(r/2)^2} = \frac{4M_i}{r}|z-w|$$
$$\le \frac{4M}{r}|z-w| < \frac{\epsilon}{\delta}\cdot\delta = \epsilon.$$

$$\left(\delta \le \frac{\epsilon r}{4M} \Rightarrow \frac{4M}{r} \le \frac{\epsilon}{\delta}\right)$$

$\square$

I'll now sketch a proof without Arzela–Ascoli, of the slightly weaker form of Montel (which is what we will mainly use):

$\boxed{\text{Proposition}}$ <u>$\mathcal{F}$ holomorphic, uniformly bounded $\Rightarrow$ $\mathcal{F}$ normal.</u>

Reductions: (1) I'll just prove that given $K \subset \Omega$ compact, any sequence has a subsequence (uniformly) convergent on K. This suffices, since the Cantor diagonal trick can be used to get uniform convergence on a nested family of compact subsets simultaneously.

(2) If we can prove the existence of a subsequence for $K = \overline{D}(\alpha, \frac{R}{2}) \subset D(\alpha, R) \subset \Omega$, this result generalizes to any compact subset by taking a finite cover by disks.

So . . .

Proof in the case $K = \overline{D}(\alpha, \frac{R}{2})$: Expand $\{f_n\}$ in a power series $f_n(z) = \sum_j a_j^{(n)} (z-\alpha)^j$ and let $\{f_{n,k}\}$ be the kth partial sums. Then since $|f_n| \le M$ in $\overline{D}(\alpha, R)$,

$$|a_j^{(n)}| = \left| \frac{f_n^{(j)}(\alpha)}{j!} \right| \overset{\text{Cauchy}}{\le} \frac{1}{2\pi} \left| \int_{\partial D(\alpha,R)} \frac{f_n(z)\,dz}{(z-\alpha)^{j+1}} \right| \le \frac{MR}{R^{j+1}} = \frac{M}{R^j}$$

$$\Longrightarrow \| a_j^{(n)} (z-\alpha)^j \|_K \le M \cdot \frac{(R/2)^j}{R^j} = \frac{M}{2^j}$$

$$\Longrightarrow \| f_n - f_{n,k} \|_K \le \frac{M/2^{k+1}}{(1 - \frac{R/2}{R})} = \frac{M}{2^k}$$

$$\Longrightarrow f_{n,k} \longrightarrow f_n \text{ uniformly on } K.$$

Using the diagonal trick and boundedness (for each fixed j) of $\{a_j^{(n)}\}_{n \in \mathbb{N}}$, one may extract $\{n_\ell\}$

such that $a_j := \lim_{l \to \infty} a_j^{(n_l)}$ exists $(\forall j)$. Pick $\epsilon > 0$ and choose:

- $k_0 \in \mathbb{N}$ s.t. $k \geq k_0 \Rightarrow \frac{M}{2^k} \leq \epsilon/3$
- $l_0 \in \mathbb{N}$ s.t. $l \geq l_0 \Rightarrow$
$$|a_j^{(n_l)} - a_j| < \frac{\epsilon}{6 \sum_{j=0}^{k_0} \left(\frac{R}{2}\right)^j} \; (\forall j \leq k_0).$$

Then for $l, l' \geq l_0$,

$$\|f_{n_l} - f_{n_{l'}}\|_K \leq \|f_{n_l} - f_{n_l,k_0}\|_K + \|f_{n_{l'}} - f_{n_{l'},k_0}\|_K + \|f_{n_l,k_0} - f_{n_{l'},k_0}\|_K$$

$$\leq \frac{M}{2^{k_0}} + \frac{M}{2^{k_0}} + \frac{\epsilon}{3\sum_{j=0}^{k_0}\left(\frac{R}{2}\right)^j} \cdot \sum_{j=0}^{k_0}\left(\frac{R}{2}\right)^j \leq \epsilon.$$

Thus the $\{f_{n_k}\}$ converge uniformly on K (to a limit function, necessarily holomorphic). $\square$

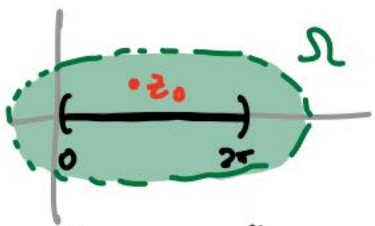
III. Further perspectives

Nothing like Montel is true in the real variable case, even for analytic functions:

Key Example // $\mathcal{F} = \{\sin(kx)\}_{k=1}^{\infty}$, $\Omega = (0, 2\pi)$.

$\mathcal{F}$ is uniformly bounded by 1. But there is not even a pointwise convergent subsequence!

To see what goes wrong, extend to complex functions on an open set Ω containing I:



$$\sin(kz) = \frac{e^{ikz} - e^{-ikz}}{2i},$$ and if we take z_0 with $\mathrm{Im}(z_0) > 0$, $|e^{-ikz_0}| \to \infty$ $(k \to \infty)$

while $|e^{ikz_0}| \to 0$.

So the "complex extension" isn't even pointwise bounded.

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Let $\mathcal{F}$ be any family of complex functions on Ω.

Define a distance function on $\mathcal{F}$ by setting

- $\delta(\alpha, \beta) := \dfrac{|\alpha - \beta|}{1 + |\alpha - \beta|}$ (for $\alpha, \beta \in \mathbb{C}$)
- $\delta_n(f, g) := \sup\limits_{z \in K_n} \delta(f(z), g(z)) = \dfrac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}$ (for $f, g \in \mathcal{F}$)

where $K_n := \{z \in \Omega \mid d(z, \Omega^c) \geq \frac{1}{n}\} \cap \overline{D}(0,n) \subset \Omega$ is a nested sequence of compact sets with $\bigcup_n K_n = \Omega$. Finally put

- $$\rho(f,g) := \sum_{n=1}^{\infty} \frac{\delta_n(f,g)}{2^n}.$$

$\boxed{\text{Proposition}}$ $\{f_m\} \subset \mathcal{F}$ converges normally (to some $f \in \overline{\mathcal{F}}$) $\iff \rho(f_m, f) \to 0 \ (m \to \infty)$.

Assuming this, we find that

$$\boxed{\begin{array}{c}\mathcal{F} \text{ is relatively compact} \\ (\text{i.e. } \overline{\mathcal{F}} \text{ is compact}) \\ \text{in the metric } \rho\end{array}} \iff \boxed{\mathcal{F} \text{ is normal}},$$

since both sides state that any sequence has a subsequence satisfying one of the equivalent conditions of the Proposition.

Proof of Prop.:

$(\Longleftarrow)$: Given $\epsilon > 0$, $\exists\, M$ s.t.

$m \geq M \Rightarrow \rho(f_m, f) < \epsilon \Longrightarrow \delta_n(f_m, f) < 2^n \epsilon.$

So $f_m \to f$ uniformly on K_n in δ_n and thus in $\|\cdot\|_{K_n}$.

$(\Longleftarrow)$: If $f_m \to f$ uniformly on each K_n, then

$\delta_n(f_m, f) \to 0$ $(\forall n)$ and thus we can

- take $N > -\log_2(\epsilon/2)$

and

- assume $m \geq M \Rightarrow \delta_n(f_m, f) < \frac{\epsilon}{2N}$ for $n \leq N$

so that

(note: $\delta_n(f_m, f) \leq 1$ for free)

$$\rho(f_m, f) = \sum_{n \geq 1} \frac{\delta_n(f_m, f)}{2^n} \leq N\left(\frac{\epsilon}{2N}\right) + \underbrace{\frac{1}{2^{N+1}}\left(1 + \tfrac{1}{2} + \cdots\right)}_{1/2^N}$$

$$< \epsilon. \qquad \square$$