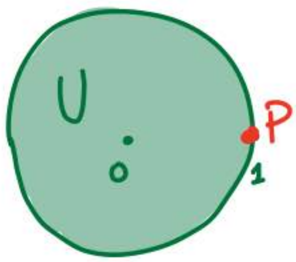
Lecture 10: The Dirichlet Problem

I. Barriers

Definition $U \subset \mathbb{C}$ open, $P \in \partial U$, $\beta : \overline{U} \to \mathbb{R}$ function.

β is a <u>barrier</u> at P $\iff$
- $\beta \in C^0(\overline{U})$
- $\beta|_U \in \underline{\mathcal{H}}(U)$
- $\beta \leq 0$
- $\{z \in \overline{U} \mid \beta(z) = 0\} = \{P\}$.

Example 1



$$\beta(z) = \underbrace{x}_{\mathrm{Re}(z)} - 1$$

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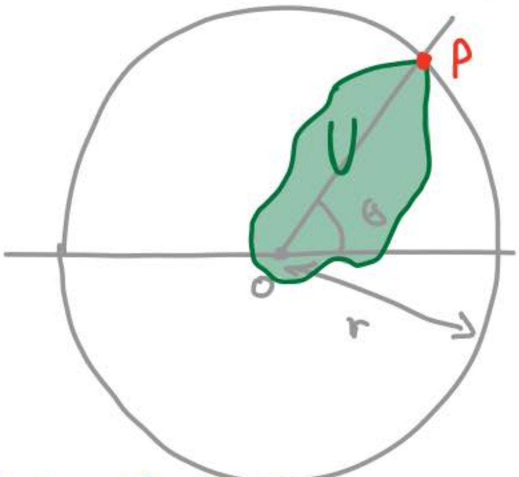
Example 2 U bounded, $P \in \partial U$ furthest point from 0 (assume this is unique)



$$\beta(z) = \mathrm{Re}(e^{-i\theta} z) - r$$

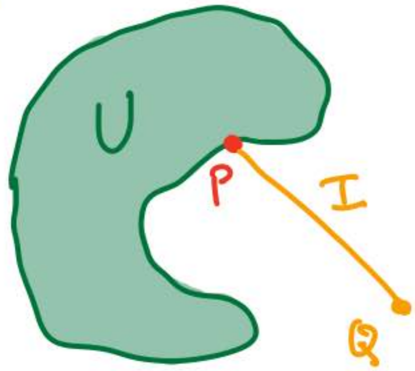
//

N.B.: $\overline{U} \subset D_r \cup \{P\}$

Example 3

Let $\psi(z) = \sqrt{\dfrac{z-P}{z-Q}} \in \mathcal{H}ol(U) \cap C^0(\overline{U})$

(also 1-1)

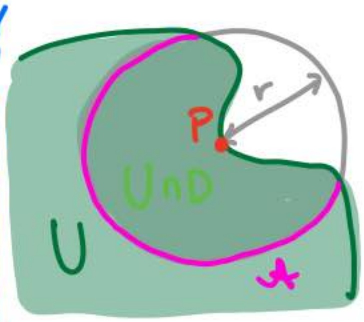


$\psi(\overline{U}) \subset$ closed $\frac{1}{2}$-plane

If φ maps this $\frac{1}{2}$-plane $\xrightarrow{\cong} D_1$, with $P \mapsto 1$, $\mathrm{Re}(\varphi \circ \psi) - 1 =: \beta$ will work.

N.B.: $I \cap \overline{U} = \{P\}$

Example 4



$p \in \partial U$, $r > 0$. If $\exists\, \beta_0$ for $U \cap D(p,r)$ at p, then $\exists\, \beta$ for U at p.

Proof: On $\overline{U \cap D(p,r)}$, the only zero of β_0 is at p (by assumption).

Put $M := \max_{q \in A} \beta_0(q)$, and let $\epsilon > 0$ be suff. small that $-\epsilon > M$. Then

$$\beta := \begin{cases} -\epsilon, & z \in \overline{U} \setminus D \cap \overline{U} \\ \max(-\epsilon, \beta_0(z)), & z \in \overline{U} \cap D \end{cases}$$

is a barrier. $\square$

Example 5 // There is no barrier for $D_1^* = U$ at $p=0$.

Proof: If there were, then

Exercise: show this is subharmonic.

$$\hat\beta(z) := \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{i\theta}z)\,d\theta$$

is also a barrier, and is constant (<0) on ∂D_1. We may assume $\hat\beta|_{\partial D_1} \equiv -1$. (Also $\hat\beta$ is rotationally invariant.)

Consider the harmonic function

$$H_r(z) := \frac{\hat\beta(r)+1}{\log r}\log|z| - 1 \quad (r\in(0,1)).$$

on D_1^*. We have $(\hat\beta - H_r)|_{C_1 \cup C_r} \equiv 0 \implies$ $\hat\beta \in \mathcal{H}$

$\hat\beta \le H_r$ on $r<|z|<1$. So we have, for every $z \in D_1^*$,

$$\hat\beta(z) \le \lim_{r\to 0^+} H_r(z) = \lim_{r\to 0^+}\left(\frac{\hat\beta(r)+1}{\log r}\log|z| - 1\right) = -1$$

$$\implies \lim_{z\to 0}\hat\beta(z) \le -1$$ contradicting that $\hat\beta \in C^0(\overline{D_1})$ and $\hat\beta(0)\,(=\hat\beta(p)) = 0$. $\Box$ //

As we will see, the existence of barriers at each boundary point of a region is closely related to the existence of solutions to the Dirichlet problem.

II. Interlude

Having done the preliminaries on subharmonic functions and barriers, we turn to Perron's solution to the Dirichlet problem and explain how — with this in hand — the 19^{th} Century analysts (starting with Riemann) had approached the Riemann mapping theorem. It was argued for a long time that the solution to Dirichlet was "heuristically clear" from electrostatics and the like. But if this is so, how should one think of this:

Example// There is no solution to the D.P. with

$$U = \mathbb{D}_1^* \text{ and } f(z) = \begin{cases} 1, & |z| = 1 \\ 0, & z = 0. \end{cases}$$

Proof: By the maximum principle, any (harmonic) solution is unique. Since the rotation of a solution is a solution, any solution F is rotationally symmetric.

So $F(z) = g(|z|) =: g(r)$, and

$$0 = \Delta F(z) = (\partial_x^2 + \partial_y^2)\, g(r)$$

$$= \partial_x r_x g'(r) + \partial_y r_y g'(r)$$

$$= (r_{xx} + r_{yy})\, g'(r) + (r_x^2 + r_y^2)\, g''(r)$$

$$= \frac{1}{r} g'(r) + g''(r)$$

$$\Rightarrow \int \frac{(g'(r))'}{g'(r)}\, dr = \int \left(-\frac{dr}{r}\right)$$

$$\Rightarrow \log(g'(r)) = -\log r + C_0$$

$$\Rightarrow g'(r) = K_0/r$$

$$\Rightarrow g(r) = A \log r + B,$$

while $g(0) = 0$ and $g(1) = 1$. $\times\!\!\times$ $\square$

This example suggests that there is "no solution" to the electrostatics problem where one has a charge at the center of the disk "different" from that on the boundary. The reason why this is a misleading way to think of the example, is that if "in the physical world" you place a finite charge at $\{0\}$, the mathematical equivalent is a δ-function at the origin — which is indeed soluble by a function of the form $A\log r + B$.

In the first section we showed D_1^* has no barrier at $\{0\}$, and indeed this will mean that the approach we develop below won't apply to this choice of U.

III. Perron's solution

We wish to prove the

Theorem Given

- $U \subset \mathbb{C}$ open / bounded, with barriers β_p $(\forall p \in \partial U)$

AND

- $f \in C^0_{\mathbb{R}}(\partial U)$,

there exists a unique $u \in C^0(\bar{U})$ with $u|_U$ harmonic and $u|_{\partial U} = f$.

more generally, the Theorem is true provided no component of ∂U is a single point — related to the Example above!

Proof: Set $\mathcal{S} := \left\{ \psi \in \underline{\mathcal{H}}(U) \,\middle|\, \limsup_{U \ni z \to p} \psi(z) \leq f(p) \ (\forall p \in \partial U) \right\}$

and note that $m := \min_{p \in \partial U} f(p)$, viewed as a constant

function, belongs to $\mathcal{S}$ (which is therefore nonempty).

We claim that

(*) $\boxed{u(z) := \sup_{\psi \in \mathcal{S}} \psi(z) \text{ solves the D.P.}}$

Step 1 $u \leq M := \max_{p \in \partial U} |f(p)|$

Otherwise there would exist $\psi \in \mathcal{S}$, $\epsilon > 0$ s.t.

$$E_\epsilon = \{ z \in U \mid \psi(z) \geq M + \epsilon \} \neq \emptyset.$$

Given $z \notin E_\epsilon$, either

- $z \in \mathbb{C} \setminus \bar{U}$ ($\Rightarrow \exists\, D(z,r) \subset \mathbb{C} \setminus \bar{U}$)
- $z \in \partial U$ (defn. of $\mathcal{S} \Rightarrow \exists\, D(z,r)$ s.t. $\psi < M+\epsilon$ on $D \cap U \Rightarrow D \subset E_\epsilon^c$)
- $z \in U \setminus E_\epsilon$ (continuity of $\psi \Rightarrow \exists\, D(z,r) \subset U \setminus E_\epsilon$).

So $\mathbb{C} \setminus E_\epsilon$ open $\Longrightarrow E_\epsilon$ closed $\underset{U \text{ bounded}}{\Longrightarrow} E_\epsilon$ compact

$\Longrightarrow \psi$ has max. (on E_ϵ, hence on U) at $q \in E_\epsilon$.

By the maximum principle for subharmonic functions,

$\psi \equiv \text{const.} \geq M + \epsilon$, contradicting $\limsup_{z \to \partial U} \psi \leq M$.

$\boxed{\text{Step 2}}$ $\underline{u \in \mathcal{H}(U)}$

Let $q \in \overline{D}(P,r) \subset U$. By defn. of u, $\exists \{\psi_j\} \subset \mathcal{S}$ with $\psi_j(q) \to u(q)$. Set

subharmonic by Lect. 6 Ex. 5

$$\Psi_n(z) := \max\{\psi_1(z), \dots, \psi_n(z)\} \in \mathcal{S}$$

subharmonic by Lect. 6 Ex. 6

$$\Phi_n(z) := \begin{cases} \Psi_n(z), & z \in U \setminus D(P,r) \\ P_{\Psi|_{\partial D(P,r)}}(z), & z \in \overset{(-)}{D}(P,r) \end{cases} \in \mathcal{S}$$

Poisson integral

minimum principle for harmonic fns.

Now $\Psi_{n+1} \geq \Psi_n \implies (\Psi_{n+1} - \Psi_n)|_{\partial D} \geq 0 \implies P_{(\Psi_{n+1}-\Psi_n)|_{\partial D}} \geq 0$
$\implies \Phi_{n+1} \geq \Phi_n$. In fact,

by defining property of subharmonic

$$\psi_n(q) \leq \Psi_n(q) \leq \Phi_n(q) \leq u(q)$$

true on all of U $\qquad$ $\Phi_n \in \mathcal{S}$

and taking $n \to \infty$ gives $\Phi_n(q) \to u(q)$ $(< \infty)$. By Harnack's principle, $\Phi_n|_D \to \Phi \in \mathcal{H}(D)$ (since $\Phi_n|_D$ are harmonic, and b/c they are increasing & bounded at q). Clearly $\Phi(q) = u(q)$.

Let $q' \in D(P,r) \setminus \{q\}$. Again, $\exists \{\rho_j\} \subset \mathcal{S}$ with $\rho_j(q') \to u(q')$; take $\tilde{\rho}_j := \max\{\rho_j, \psi_j\} \in \mathcal{S}$ and $\Lambda_n := \max\{\tilde{\rho}_1, \dots, \tilde{\rho}_n\} \in \mathcal{S}$. Clearly $\Lambda_n \leq \Lambda_{n+1}$ and $\Lambda_n(q') \to u(q')$. Set

$$H_n(z) := \begin{cases} \Lambda_n(z), & z \in U \setminus D(P,r) \\ P_{\Lambda_n|_{\partial D(P,r)}}, & z \in \overset{(-)}{D}(P,r). \end{cases}$$

Once again, $H_n \le H_{n+1}$, $\Lambda_n(q') \le H_n(q') \le u(q')$, and so

$H_n(q') \to u(q') \underset{\text{Harnack}}{\Longrightarrow} H_n|_D \to H \in \mathcal{H}(D)$, with $H(q') = u(q')$.

But also $\Phi_n \le H_n \le u$ on $U \Rightarrow \Phi \le H \le u|_D$

on $D \Rightarrow u(q) = \Phi(q) \le H(q) \le u(q) \Rightarrow H(q) = u(q)$.

In fact, the limit defines $\tilde{H}, \tilde{\Phi} \in C^0(\overline{D})$ with $\tilde{H}|_D = H$,

$\tilde{\Phi}|_D = \Phi$, and $\tilde{\Phi} \le \tilde{H} \Longrightarrow (\tilde{\Phi} - \tilde{H})|_{\partial D} \le 0$. But then

$(\Phi - H)(q) = u(q) - u(q) = 0 \xrightarrow[\text{for harmonic}]{\text{max. princ.}} \tilde{\Phi} - \tilde{H} \equiv \text{const.} = 0$

$\Longrightarrow u(q') = H(q') = \Phi(q') \underset{q \text{ was arbitrary}}{\Longrightarrow} u|_D \equiv \Phi \; (\in \mathcal{H}(D))$.

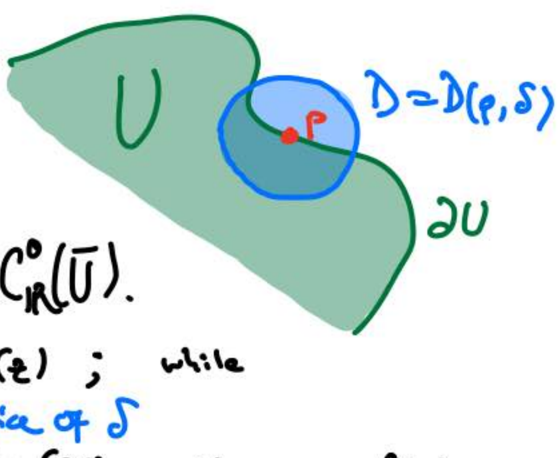
$\underset{D \text{ was arbitrary}}{\Longrightarrow} u \in \mathcal{H}(U)$.

$\boxed{\text{Step 3}}$ $\underline{\lim_{U \ni z \to p} u(z) = f(p) \quad (\forall p \in \partial U)}$ (completing the proof)

Let $p \in \partial U$, $\epsilon > 0$. U bounded $\Longrightarrow \partial U$ compact $\Longrightarrow$

f uniformly $C^0 \Longrightarrow \exists\, \delta > 0$ s.t. $|p_1 - p_2| < \delta \Rightarrow |f(p_1) - f(p_2)| < \epsilon$.

Set $\mu := \min_{z \in \overline{U} \setminus D(p,\delta)} \beta_p(z) \; (< 0)$,



$g(z) := f(p) + \underbrace{\frac{\beta_p(z)}{\mu}}_{\text{(subharm., } \le 0)} \overbrace{(M - f(p))}^{\le 0} + \epsilon$

$\in \overline{\mathcal{H}}(U) \cap C^0_{\mathbb{R}}(\overline{U})$.

On $D \cap \partial U$, $g(z) \ge f(p) + \epsilon \underset{\text{choice of } \delta}{>} f(z)$; while

on $\partial U \setminus (D \cap \partial U)$, $g(z) \underset{\text{defn. of } \mu:\ \frac{\beta_p(z)}{\mu} \ge 1}{\ge} f(p) + \epsilon + (M - f(p)) = M + \epsilon > f(z)$.

But then $\Psi \in \mathcal{S} \Rightarrow \Psi - g \in \underline{\mathcal{H}}(U)$ and

$$\limsup_{U \ni z \to p'} (\Psi - g) \leq f(p') - f(p') = 0 \quad (\forall p' \in \partial U) \xRightarrow[\text{for } \underline{\mathcal{H}}]{\text{max. princ.}}$$

$$\Psi \leq g \text{ in } U \xRightarrow[\text{on LHS}]{\text{take sup}} u \leq g \text{ in } U$$

$$\Longrightarrow \limsup_{U \ni z \to p} u(z) \leq g(p) = f(p) + \epsilon.$$

Next consider $\tilde{g}(z) := f(p) - \underbrace{\frac{\overset{\leq 0}{\beta_p(z)}}{\mu}(M + f(p))}_{\leq 0} - \epsilon \in \underline{\mathcal{H}}(U) \cap C^0_{\mathbb{R}}(\bar{U})$.

On $D \cap \partial U$, $\tilde{g}(z) \leq f(p) - \epsilon \overset{\text{choice of } \delta}{<} f(z)$; while

on $\partial U \setminus (D \cap \partial U)$, $\tilde{g}(z) \leq f(p) - (M + f(p)) - \epsilon = -M - \epsilon < f(z)$.

Hence $\tilde{g} \in \mathcal{S} \Rightarrow \tilde{g} \leq u$ on $U \Rightarrow \liminf_{U \ni z \to p} u(z) \geq \tilde{g}(p) = f(p) - \epsilon$.

Putting both inequalities together,

$$f(p) - \epsilon \leq \liminf_{z \to p} u(z) \leq \limsup_{z \to p} u(z) \leq f(p) + \epsilon$$

and taking $\epsilon \to 0$ forces $\lim_{U \ni z \to p} u(z) = f(p)$. Since $p \in \partial U$ was arbitrary, we are done. $\square$

Remark/ In $\S$I, we saw that barriers exist for all $p \in \partial U$ provided each component of ∂U is continuously differentiable. //