Lecture 11: Applications of Dirichlet

I. Dirichlet $\Rightarrow$ RMT!

Well, for ∂U = single C^1 curve at any rate.
(Recall that a shortcoming of the approach via Dirichlet is the messiness of dealing with the boundary.)

So assume $U \subseteq \mathbb{C}$ is open, bounded, connected, & simply connected with C^1 Jordan boundary curve. Let $P \in U$ be the point we want to send to $0 \in D_1$, and set

$$\Gamma(z) := \log|z-P| \in \mathcal{H}(\bar{U}\setminus\{P\})$$

$$\mu := \text{Solution to D.P. with boundary data } (-\Gamma|_{\partial U})$$

$$G := \Gamma + \mu \in \mathcal{H}(U\setminus\{P\}) \cap C^0_{\mathbb{R}}(\bar{U}\setminus\{P\})$$ "Green's fcn. for U"

Let $\gamma \subset U\setminus\{P\}$ be a closed C^1 path, recall $\gamma \underset{\text{hom}}{\equiv} W(\gamma,P)\cdot\partial D(P,\epsilon)$ so that

$$(\ast\ast)\qquad \oint_\gamma \underbrace{\left\{-\frac{\partial G}{\partial y}dx + \frac{\partial G}{\partial x}dy\right\}}_{=:\,\omega \text{ (closed differential)}^\dagger} = W(\gamma,P)\oint_{\partial D(P,\epsilon)} \omega$$

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† really the harmonic conjugate of the differential of G. We note that $d(-G_y\,dx + G_x\,dy) = -G_{yy}\,dy\wedge dx + G_{xx}\,dx\wedge dy = (G_{xx}+G_{yy})\,dx\wedge dy = 0$ since $0 = \Delta G = G_{xx}+G_{yy}$; so ω is indeed closed (on $U\setminus\{P\}$).

Clearly the "μ" part doesn't contribute to this integral, since $-\frac{\partial\mu}{\partial y}dx + \frac{\partial \mu}{\partial x}dy$ is closed on U. So (**) becomes

$$= W(\gamma, P) \oint_{\partial D(P,\epsilon)} \left\{ -\frac{\partial \Gamma}{\partial y}dx + \frac{\partial \Gamma}{\partial x}dy \right\}$$

note: $\frac{\partial \Gamma}{\partial x} = \frac{x - x_p}{r^2}$, $\frac{\partial \Gamma}{\partial y} = \frac{y-y_p}{r^2}$

$$= W(\gamma, P) \int_0^{2\pi} \left\{ -\frac{\sin\theta}{\epsilon} d(\epsilon\cos\theta) + \frac{\cos\theta}{\epsilon} d(\epsilon \sin\theta) \right\}$$

$$= W(\gamma, P) \int_0^{2\pi} \{ \underbrace{\sin^2\theta + \cos^2\theta}_{1} \} d\theta$$

$$= 2\pi W(\gamma, P) \quad \in \quad 2\pi \mathbb{Z}$$

Hence, $f(z) := \exp\left\{ G(z) + i \int_{P_0\,(P\notin)}^{z} \left(-\frac{\partial G}{\partial y}dx + \frac{\partial G}{\partial x}dy \right) \right\}$

is well-defined and holomorphic on $U\setminus\{P\}$. Now write

$$d\theta = d \arctan \frac{Y}{X} = \frac{-Y/X^2}{1+Y^2/X^2}dX + \frac{1/X}{1+Y^2/X^2}dY = -\frac{Y}{R^2}dX + \frac{X}{R^2}dY$$

and observe from this that $-\frac{\partial \Gamma}{\partial y}dx + \frac{\partial \Gamma}{\partial x}dy = d\arg(z-P)$ so that the factor

$$\exp\left\{ \Gamma(z) + i\int_{P_0}^{z}\left(-\frac{\partial\Gamma}{\partial y}dx + \frac{\partial \Gamma}{\partial x}dy\right)\right\} = \exp\{\log|z-P| + i\arg(z-P)\}$$

$$= \exp\{\log(z-P)\} = z - P$$

is also holomorphic $\Rightarrow \begin{cases} f(z) \in \mathrm{Hol}(U) \\ f(P) = 0 \quad \text{to \underline{first} order (simple zero).} \end{cases}$

Also, since $G|_{\partial U} \equiv 0$, as $z_i \to \partial U$ f assumes the form

$\exp(0 + i\mathbb{R}) \Rightarrow |f(z_i)| \to 1 \xRightarrow[\text{for holo.}]{\text{MMP}} |f| \leq 1$ on U

$\Longrightarrow f(U) \subseteq D_1$.

Now clearly P is the <u>only</u> zero of f in U (the argument of the exp doesn't blow up anywhere else). More generally, for any $\xi \in D_1$, choose a closed path $\gamma \subset U$ so that $f(\gamma) \subset D_1 \setminus D_{1-\epsilon}$ for $1-\epsilon > |\xi|$ (why can we do this?); then we have

$$\#\left\{\begin{matrix}\text{zeroes (w/multiplicity)} \\ \text{of } f-\xi \text{ in } U\end{matrix}\right\} = \int_\gamma \frac{f'}{f-\xi}\,dz$$

$(w = f(z))$

$$= \int_{f(\gamma)} \frac{dw}{w-\xi}$$

$$= W(f(\gamma), \xi)$$

(use $f(\gamma) \subset D_1 \setminus D_{1-\epsilon}$ again)

$$= W(f(\gamma), 0)$$

$$= \int_{f(\gamma)} \frac{dw}{w}$$

$$= \int_\gamma \frac{f'}{f}\,dz$$

$$= \#\left\{\begin{matrix}\text{zeroes (w/multiplicity)} \\ \text{of } f \text{ in } U\end{matrix}\right\}$$

$$= 1.$$

So f is 1-to-1 and onto — it assumes the value ξ exactly once. This proves RMT.

In HW #4 you'll carry out a similar procedure for a non-simply-connected region, using the "period" of a harmonic conjugate of "$d\theta$". Speaking of which:

II. Conformal equivalence of annuli

Proposition Let $A_i = \{z \in \mathbb{C} \mid 1 < |z| < R_i\}$ $(i=1,2)$. Then $A_1 \cong A_2$ (conformal isomorphism) $\Leftrightarrow R_1 = R_2$.

Proof ($\Rightarrow$, of course): Suppose $\phi : A_1 \xrightarrow{\cong} A_2$ (holo.).

As we know, if $z_i \to \partial A_1$, then $\phi(z_i) \to \partial A_2$.

In particular, $\phi^{-1}\{|w| = \frac{1+R_2}{2}\}$ is compact and avoids $\{1 < |z| < 1+\epsilon\}$ for $\epsilon > 0$ sufficiently small. Any sequence $z_i \to \{|z|=1\} \subset \partial A_1$ has the property that (for i suff. large) z_i and z_{i+1} can be connected by a path $\mathcal{P} \subset \{1 < |z| < 1+\epsilon\}$, and $\phi(\mathcal{P})$ avoids $\{|w| = \frac{1+R_2}{2}\}$. So $\phi(z_i)$ approaches either $\{|w|=1\}$ or $\{|w|=R_2\}$ but not both. Also, we cannot have that $z_i \to \{|z|=1\}$ and $z_i' \to \{|z| = R_1\}$ both yield $\phi(z_i), \phi(z_i') \to \{|w|=1\}$ or $\{|w|=R_2\}$: otherwise, by the MMP, ϕ would be constant. The upshot is that, by inversion if necessary, we may assume that

$$(**) \quad \boxed{\begin{array}{l} z_i \to \{|z|=1\} \Rightarrow \phi(z_i) \to \{|w|=1\} \\ z_i \to \{|z|=R_1\} \Rightarrow \phi(z_i) \to \{|w|=R_2\}. \end{array}}$$

Consider next

$$h(z) := \log|z| \log R_2 - \log|\phi(z)| \log R_1 \in \mathcal{H}(A_1) \cap C^0_{\mathbb{R}}(\bar{A_1}).$$

Clearly $|z_i| \to 1 \Rightarrow h(z_i) \to 0$

and $|z_i| \to R_1 \Rightarrow h(z_i) \to 0$.

So $h|_{\partial A_1} \equiv 0 \xRightarrow[\text{principle}]{\text{max.}} h \equiv 0 \Rightarrow \log|\phi(z)| = \frac{\log R_2}{\log R_1} \log|z|$

$$\xRightarrow[\exp]{} |\phi(z)| = |z|^\beta, \quad \beta := \frac{\log R_2}{\log R_1}.$$

Now on $D(P,r) \subset A_1$ (but not on all of A_1),

$z^\beta := \exp\{\beta \underbrace{\log z}\}$ is well-defined. So

branch of $\log(z)$
is well-def'd. on $D(P,r)$

$\left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)}$ is well-defined, with $\left| \left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)} \right| = 1$

$\xRightarrow[\text{OMT}]{} \left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)} \equiv \text{const.} \Rightarrow \phi(z)|_{D(P,r)} = \alpha z^\beta, \quad |\alpha| = 1.$

As $D(P,r)$ was arbitrary, ϕ gives a "branch" of z^β

on $\underline{\text{all of } A_1}$ $\Longrightarrow$ $\beta \in \mathbb{Z}$. By (**), $\beta > 0$,
and since ϕ is 1-to-1, $|\beta| = 1$. So $\beta = 1$ and
$\phi(z) = \alpha z$ for some $|\alpha| = 1$ $\Longrightarrow$ $R_2 = R_1$. $\square$