Lecture 12: Multiply connected regions (I)

The connectivity of a region $U \subset \mathbb{C}$ is the number of connected components of U^c. So simply connected regions have connectivity $n = 1$. Before stating any general results for $n > 1$, we first look at $n = 2$. In HW#4 you will use a procedure similar to that in Lecture 11 to map somewhat arbitrary regions of connectivity $n = 2$ biholomorphically to annuli. Here, as a warmup, I'm going to prove something just for annuli.

I. Conformal equivalence of annuli

$\boxed{\text{Proposition}}$ Let $A_i = \{z \in \mathbb{C} \mid 1 < |z| < R_i\}$ $(i = 1, 2)$. Then $A_1 \cong A_2$ (conformal isomorphism) $\Longleftrightarrow R_1 = R_2$.

Proof ($\Rightarrow$, of course): Suppose $\phi : A_1 \xrightarrow{\cong} A_2$ (holo.).

As we know, if $z_i \to \partial A_1$, then $\phi(z_i) \to \partial A_2$. In particular, $\phi^{-1}\{|w| = \frac{1+R_2}{2}\}$ is compact and avoids $\{1 < |z| < 1+\epsilon\}$ for $\epsilon > 0$ sufficiently small. Any sequence

$z_i \to \{|z|=1\} \subset \partial A_1$ has the property that (for i suff. large) z_i and z_{i+1} can be connected by a path $\mathcal{P} \subset \{1<|z|<1+\epsilon\}$, and $\phi(\mathcal{P})$ avoids $\{|w| = \frac{1+R_2}{2}\}$. So $\phi(z_i)$ approaches either $\{|w|=1\}$ or $\{|w|=R_2\}$ but not both. Also, we cannot have that $z_i \to \{|z|=1\}$ and $z_i' \to \{|z|=R_1\}$ both yield $\phi(z_i), \phi(z_i') \to \{|w|=1\}$ or $\{|w|=R_2\}$: otherwise, by the MMP, ϕ would be constant. The upshot is that, by inversion if necessary, we may assume that

$$(**) \quad \boxed{\begin{array}{l} z_i \to \{|z|=1\} \Rightarrow \phi(z_i) \to \{|w|=1\} \\ z_i \to \{|z|=R_1\} \Rightarrow \phi(z_i) \to \{|w|=R_2\}. \end{array}}$$

Consider next

$$h(z) := \log|z| \log R_2 - \log|\phi(z)| \log R_1 \in \mathcal{H}(A_1) \cap C^0_{\mathbb{R}}(\bar{A}_1).$$

Clearly $|z_i| \to 1 \Rightarrow h(z_i) \to 0$

and $|z_i| \to R_1 \Rightarrow h(z_i) \to 0$.

$$\text{So } h|_{\partial A_1} \equiv 0 \xRightarrow[\text{principle}]{\text{max.}} h \equiv 0 \Rightarrow \log|\phi(z)| = \frac{\log R_2}{\log R_1} \log|z|$$

$$\xRightarrow[\exp]{} |\phi(z)| = |z|^\beta, \ \beta := \frac{\log R_2}{\log R_1}.$$

Now on $D(P,r) \subset A_1$ (but not on all of A_1),

$z^\beta := \exp\{\beta \underbrace{\log z}_{\text{branch of } \log(z) \text{ is well-def'd. on } D(P,r)}\}$ is well-defined. So

$\left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)}$ is well-defined, with $\left|\left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)}\right| \equiv 1$

$\underset{\text{OMT}}{\Longrightarrow} \left.\frac{\phi(z)}{z^\beta}\right|_{D(P,r)} \equiv \text{const.} \Longrightarrow \phi(z)\Big|_{D(P,r)} = \alpha z^\beta, \quad |\alpha| = 1.$

As $D(P,r)$ was arbitrary, ϕ gives a "branch" of z^β

on <u>all of A_1</u> $\Longrightarrow \beta \in \mathbb{Z}$. By (**), $\beta > 0$,

and since ϕ is 1-to-1, $|\beta| = 1$. So $\beta = 1$ and

$\phi(z) = \alpha z$ for some $|\alpha| = 1 \Longrightarrow R_2 = R_1$. $\square$

II. Dirichlet for multiply connected regions

Generalizing the result just established, we now discuss canonical mappings and conformal invariants for regions with connectivity $n > 1$, following Ahlfors rather closely.

The two main tools will be harmonic measures and Green's functions. We begin with a (partial) reformulation of what was proved regarding the Dirichlet problem:

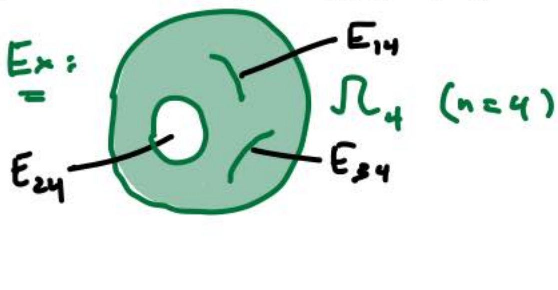
$\boxed{\text{Theorem 1}}$ Let $\Omega \subset \mathbb{C}$ be a region, with $\mathbb{C} \setminus \Omega$ having a finite number of connected components, none of which are points. Then the Dirichlet problem is soluble for Ω.

(in case $\partial\Omega$ is a union of C^0 Jordan curves)

Proof: Write $\mathbb{C} \setminus \Omega = E_1 \amalg \cdots \amalg E_n$, with E_n the unbounded component. By the RMT, there exists a conformal equivalence $\phi_n : \mathbb{C} \setminus E_n \xrightarrow{\cong} D_1$. Set $\Omega_n := \phi_n(\Omega)$,

$$E_{kn} := \begin{cases} \mathbb{C} \setminus D_1, & k = n \\ \phi_n(E_k), & k < n. \end{cases}$$

Ex:



Next, consider $\hat{\mathbb{C}} \setminus E_{n-1,n}$, which is simply connected, so RMT $\Rightarrow \exists\, \phi_{n-1} : \hat{\mathbb{C}} \setminus E_{n-1,n} \xrightarrow{\cong} \hat{\mathbb{C}} \setminus \overline{D_1}$; we set

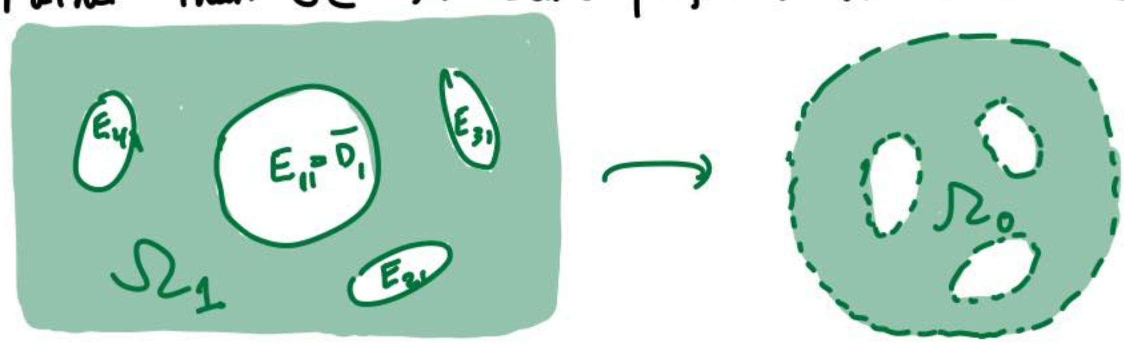
$\infty \longmapsto \infty$

$\Omega_{n-1} := \phi_{n-1}(\Omega_n)$, $E_{k,n-1} := \begin{cases} \overline{D}_1, & k = n-1 \\ \phi_{n-1}(E_{kn}), & k \neq n-1 . \end{cases}$

Iterating this last step for $n-2, n-3, \ldots, 1$ we see that for $k \geq l$, ∂E_{kl} is real-analytic Jordan by virtue of being the image of ∂D_1 by a composition of conformal 1-to-1 maps. In particular, $\partial \Omega_1 =: C = \sum_{i=1}^{n} C_i \; (=: \partial E_{i1})$ is a union of such curves and so Ω_1 has a barrier at any point of $\partial\Omega_1$. Pulling these barriers back by $\phi_1 \circ \cdots \circ \phi_n : \overline{\Omega} \to \overline{\Omega_1}$ (extend by Caratheodory) shows the Dirichlet problem is soluble for Ω. $\square$

Let Ω_0 be the image of Ω_1 via $z \mapsto \frac{1}{z}$.

In what follows it is convenient to work with $\tilde{\Omega} := \Omega_0$ rather than Ω for some purposes. Write $\partial\Omega = \sum \partial_i \Omega$.



III. Harmonic measures

Definition 1 The harmonic measure of $\partial_k \Omega$ with respect to Ω is the solution to the D.P. with $f = \begin{cases} 1 & \text{on } \partial_k \Omega \\ 0 & \text{on } \partial_\ell \Omega,\ \ell \neq k. \end{cases}$

Call this ω_k ($\in \mathcal{H}(\Omega)$).

PROPERTIES:

1. Partition of unity: $\sum_{k=1}^{n} \omega_k = 1$.

   Proof: Use max-min principle for harmonic functions. □

2. Continuous extension to boundary:

$$\begin{array}{ccc} \omega_\ell : \overline{\Omega} & \to & [0,1] \\ \cup & & \cup \\ \Omega & \to & (0,1) \end{array}$$

   Proof: range is confined to $[0,1]$ again by max-min principle. □

Now we restrict to the "smooth boundary case" — i.e., work with $\tilde{\Omega}$. I won't change the notation for the ω_k's.

3. Schwarz extension: Using technique of reflection about an analytic arc (when the harmonic function has C^0 extension to this arc and is 0 (or constant) on the arc), we obtain $\tilde{\omega}_\ell \in \mathcal{H}(\overline{\tilde{\Omega}})$ — i.e. a harmonic extension on a neighborhood of $\overline{\tilde{\Omega}}$, with $\tilde{\omega}_\ell|_{\tilde{\Omega}} = \omega_\ell$. (This is false for Ω — it would violate max/min on a region w/slits)

Definition 2 The <u>periods</u> of Ω are the real numbers

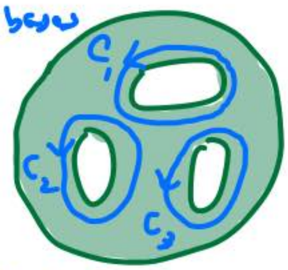
$$\alpha_k^j := \int_{C_k} * d\tilde{\omega}_j \,, \quad k, j = 1, \dots, n-1.$$

(4) <u>Conformal invariance</u>: If $\Omega \cong \Omega'$ conformally, then after reordering indices so that the $\partial_i \Omega$ and $\partial_i \Omega'$ match up, $`\alpha_k^j = \alpha_k^j$ $(\forall j, k)$.

Proof: Get conformal isom. $\tilde{\Omega} \to \tilde{\Omega}'$, pull everything back — pullback of harmonic by holo. is harmonic, and $d, *$, & periods are all invariant under holo. pullback. $\square$

(5) <u>Linear independence</u>: The vectors $\left\{ \underline{\alpha}^j = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_{n-1}^j \end{pmatrix} \right\}_{j=1,\dots,n-1} \subset \mathbb{R}^{n-1}$ are a basis.

Remark// For what? Well, we have bases



$\tilde{\Omega}$

$\{C_k\}_{k=1}^{n-1} \subset H_1(\bar{\tilde{\Omega}}, \mathbb{Z})$

and $\{C_k^\vee\}_{k=1}^{n-1} \subset H^1(\bar{\tilde{\Omega}}, \mathbb{Z})$ (or $\mathbb{R}$); while

$$H^1(\bar{\tilde{\Omega}}, \mathbb{R}) \cong H^1_{dR}(\tilde{\Omega}) = \frac{\text{closed real 1-forms}}{\text{exact real 1-forms}}.$$

Further, the vector $\begin{pmatrix} \int_{C_1} * d\tilde{\omega}_j \\ \vdots \\ \int_{C_{n-1}} * d\tilde{\omega}_j \end{pmatrix}$ expresses $* d\omega_j$ with respect to the basis $\{C_k^\vee\}$, so the above result says that $\{* d\omega_j\}_{j=1}^{n-1}$ is a basis for de Rham cohomology. //

Proof: Suppose $\sum \lambda_j \underline{\alpha}^j = 0$ for some $\{\lambda_j\} \subset \mathbb{R}$, i.e. $* d(\sum \lambda_j \omega_j)$ has trivial periods. Then

$$f := \sum \lambda_j \omega_j + i \int * d(\sum \lambda_j \omega_j) \in \mathcal{H}ol(\Omega)$$

is well-defined, with $\mathrm{Re}(f)\big|_{C_\ell} \equiv \begin{cases} \lambda_\ell, & \ell = 1, \dots, n-1 \\ 0, & \ell = n. \end{cases}$

Assume f nonconstant, so that $\exists\, w_0$ with $\mathrm{Re}(w_0) \neq \begin{cases} 0 \text{ or} \\ \text{any } \lambda_\ell \end{cases}$, and $z_0 \in \tilde{\Omega}$ with $f(z_0) = w_0$ (using OMT). By the argument principle

$$0 < \int_{\partial \tilde{\Omega}} d\arg(f(z) - w_0) = \sum_\ell \int_{C_\ell} d\arg(f(z) - w_0)$$
$$= \sum_\ell \int_{f(C_\ell)} d\arg(w - w_0), \quad (*)$$

where $f(C_\ell)$ is contained in a vertical strip on $\mathrm{Re}(w) = \begin{cases} \lambda_\ell, & \ell \neq n \\ 0, & \ell = n. \end{cases}$ But then $(*) = 0$, a contradiction. So f is constant, and all $\lambda_\ell = 0$. $\square$

(6) Symmetry: $\alpha_k^j = \alpha_j^k$.

Proof: Exercise using Theorem 19 of Chap. 4 of Ahlfors. $\square$

Next time we shall use property 5 to prove

Theorem 2 $\exists\, \phi : \Omega \xrightarrow[\text{conformal}]{\cong} \underbrace{A(1, r_1)}_{\text{annulus}} \setminus \coprod_{i=2}^{n-1} \underbrace{C_{r_i}(\theta_i^a, \theta_i^b)}_{\text{arcs}}$