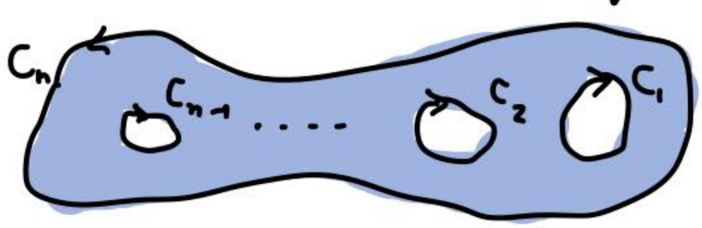
Lecture 13: Multiply connected regions (II)

Last time we introduced harmonic measures ω_j and associated periods α_k^j, and proved several properties of them. (See Lecture 12.) The main point is the following application.

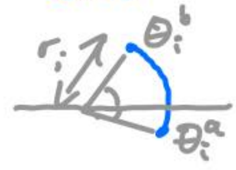
I. A "mapping theorem" for multiply connected regions

Recall the setting: $\Omega \subset \mathbb{C}$ is a region for which the Dirichlet problem is soluble, and $\tilde{\Omega} \xleftarrow{\sim} \Omega$ a biholomorphic one with C^1 Jordan boundary components. We write $\partial\tilde{\Omega} = \sum_{i=1}^{n} C_i$, with C_n the "outer" boundary component.



$\boxed{\text{Theorem 2}}$ $\exists\, \phi : \Omega \xrightarrow[\text{conformal}]{\cong} \underbrace{A(1, r_1)}_{\text{annulus}} \setminus \coprod_{i=2}^{n-1} \underbrace{C_{r_i}(\theta_i^a, \theta_i^b)}_{\text{arcs}}$

Proof: Write $\underline{\underline{\alpha}} = [\alpha_k^j]$ $\quad$ ((n-1) x (n-1) matrix) $\quad$ $j \downarrow$, $\xrightarrow{k}$



By ⑤ there is a solution to ${}^t\underline{\lambda} \cdot \underline{\underline{\alpha}} = (2\pi, 0, \ldots, 0)$, and taking $f = \sum_{j=1}^{n-1} \lambda_j \omega_j + i \int \sum_j \lambda_j (\star d\omega_j)$

(= multivalued function with periods $2\pi i$ about C_1 and $-2\pi i$ about C_n)

we find $F := e^f \in Hol(\bar{\Omega})$ (using property ③).

Clearly
$$\underset{e^{Re(f)}}{|F|}\Big|_{C_k} \equiv \begin{cases} e^{\lambda_k}, & k \neq n \\ 1, & k = n \end{cases}, \quad \text{and we put}$$

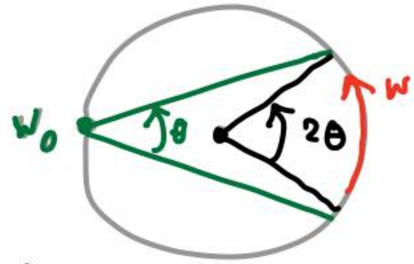
$r_i := e^{\lambda_i}$ $(i = 1, \ldots, n-1)$. Since $\partial\bar{\Omega} = \sum_{i=1}^{n} C_i$, by the argument principle together with the fact that $\arg(w - w_0)$ jumps by π when w passes over w_0,

$$(**) \quad \boxed{\frac{1}{2\pi}\sum_{i=1}^{n}\int_{C_i} d\arg(F(z) - w_0) = \#\{z \in \bar{\Omega} \mid F(z) = w_0\} + \tfrac{1}{2}\#\{z \in \partial\bar{\Omega} \mid F(z) = w_0\}}$$

where both "#'s" are interpreted with multiplicity.

To compute LHS(**), use the fact that the $F(C_i)$ are circular arcs. For w, w_0 on the <u>same</u> circular arc, we have

$\left({}^{**}_{\ *}\right)$ $\boxed{d\arg(w-w_0) = \frac{1}{2}\, d\arg(w).}$



We know that $\arg(F(z)) = \mathrm{Im}(f(z))$ has periods $\begin{cases} \pm 2\pi & \text{about } C_1 \,\&\, C_n \\ 0 & \text{about } C_2, \dots, C_{n-1} \end{cases}$ by construction.

So (a) $F(C_1)$ and $F(C_n)$ are full circles

(b) C_1 & C_n are mapped in 1-1 fashion on these circles, because LHS $(**)$ is (using $\left({}^{**}_{\ *}\right)$ and (a))

$\begin{cases} -\frac{1}{2}+1 = \frac{1}{2} & \text{for } w_0 \in F(C_n) \\ 0+\frac{1}{2} = \frac{1}{2} & \text{for } w_0 \in F(C_1) \end{cases}$, so that RHS $(**)$ is $0 + \frac{1}{2}\cdot\boxed{1}$.

(c) $C_2, \dots, C_{n-1}$ are mapped to a single <u>back-and-forth</u> along a slit, as $w_0 \in F(C_l)_{\,l=2,\dots,n-1}$ yields 1 on LHS $(**)$, which on RHS $(**)$ must be interpreted as $0 + \frac{1}{2}\cdot\boxed{2}$ $(=1)$ (as $1 + \frac{1}{2}(0)$ is impossible).

In fact, this numerical condition shows that $F(\partial\tilde{\Omega}) \cap F(\tilde{\Omega})$ is empty; and since $F(\tilde{\Omega})$ is connected, <u>this</u> is why $F(C_l)$ $(l = 2, \dots, n-1)$ must be contained in <u>slits</u> (= <u>proper</u> connected closed subsets of circles). $\square$

By uniqueness of the choice of F (up to $f \mapsto f + C \Rightarrow F \mapsto F \cdot K$), we get uniqueness of the image region $F(\tilde{\Omega})$ up to rotation. (Possible dilations are taken care of by the normalization of the inner radius.) This gives the

$\boxed{\text{Corollary}}$ <u>An n-connected region Ω has</u>

$$\underline{\overset{r_1}{1} + 3\overset{r_\ell,\, \theta^a_\ell,\, \theta^b_\ell\ (\ell = 2, \dots, n-1)}{(n-2)} - \overset{\text{rotation}}{1} = 3n-6}$$

<u>real moduli — i.e., is determined up to conformal equivalence by the point</u>

$$\underline{\left[(r_1, r_2, \dots, r_{n-1};\ \theta^a_2, \dots, \theta^a_{n-1};\ \theta^b_2, \dots, \theta_2^{n-1})\right] \Big/ \left(\underset{\equiv}{\text{rotations}}\right)}$$

<u>in $\mathbb{R}_+^{n-1} \times \dfrac{(\mathbb{R}/2\pi\mathbb{Z})^{2n-4}}{\mathbb{R}/2\pi\mathbb{Z}}$.</u>

there should also be action of $\mathfrak{S}_n$ here (see the postscript).

One might ask: <u>is this point determined by the periods $\{\alpha^i_k\}$?</u> In this case they would furnish a complete set of conformal invariants. While we'll see in §III that this follows from a result in algebraic geometry, it would be interesting to have a direct proof. For now, that this <u>should</u> be true is strongly suggested by the following table:

(accounting for symmetry relations $d^j_k = d^k_j$)

n (= connectivity)	$\binom{n}{2}$ (= # of independent $\{d^j_k\}$)	$\begin{cases} 3n-6, & n>2 \\ 1, & n=2 \end{cases}$ (# of moduli)
2	1	1
3	3	3
4	6	6
5	10	9
6	15	12
$\vdots$	$\vdots$	$\vdots$

Which also says that there eventually appears a <u>locus</u> in the space of periods — the period $\binom{n}{2}$-tuples that "can come from a region". <u>What is this locus</u>? I don't know if this is totally understood. These two questions are "real" instances of the <u>Torelli theorem</u> and <u>Schottky problem</u> for algebraic curves (a.k.a. Riemann surfaces). Here "real" refers to the fact that the differentials/periods in the latter case are complex, so the situation is a bit different.

$\boxed{\text{II. Green's functions}}$

Definition 3 Given a region $\Omega \subset \mathbb{C}$, a <u>Green's function</u> of Ω with singularity at z_0 is a function

$$g(z, z_0) : \Omega \setminus \{z_0\} \longrightarrow \mathbb{R}$$

with the properties

(a) $g(z, z_0)$ is harmonic in $\Omega \setminus \{z_0\}$

(b) $G(z) := g(z, z_0) + \log |z - z_0|$ is harmonic in a disk about z_0

(c) $\lim_{\substack{z \to \zeta \\ \Omega \ni z}} g(z, z_0) = 0 \quad \forall \zeta \in \partial\Omega$.

($\Rightarrow$ yields a C^0 function on $\overline{\Omega} \setminus \{z_0\}$ too).

4 EASY PROPERTIES:

(1) Uniqueness: Given $g, \tilde{g}$ satisfying (a)-(c) for Ω, z_0,

$$\left.\begin{array}{l} \text{(a)-(b)} \Rightarrow g - \tilde{g} \in \mathcal{H}(\Omega) \\ \text{(c)} \Rightarrow g - \tilde{g} \xrightarrow[z \to \partial\Omega]{} 0 \end{array}\right\} \xRightarrow[\text{max/min}]{} g - \tilde{g} = 0.$$

(2) Existence: If $\Omega \subset \mathbb{C}$ is bounded, and the D.P. is solvable in Ω, then for each z_0 $\exists$ Green's function: just solve D.P. for $\log|z-z_0|\,\big|_{\partial\Omega} \rightsquigarrow G(z)$, then subtract off $\log|z-z_0|$.

Remark: Green's function does NOT exist for, say, $\Omega = \mathbb{C}$. (The problem, naturally, is the lack of a conformal mapping into a bounded region.) I'll leave it as an exercise — use the following property:

(3) Positivity: $g > 0$ on $\Omega \setminus \{z_0\}$.

Clearly this is true sufficiently close to $\{z_0\}$, since $\underbrace{\log|z-z_0|}_{\to -\infty} + g(z, z_0)$ is bounded. Applying (c) and max/min on $\Omega \setminus \overline{D}(z_0, \epsilon)$ does the job.

④ Invariance: If $\phi : \Omega \xrightarrow{\cong} \Omega'$ $(z_0 \longmapsto z_0')$

is a conformal isomorphism, then $g(\phi(z), z_0') = g(z, z_0)$

— i.e. the pullback of a Green's function yields a Green's function.

Sketch: The main point is that $\log|\underbrace{\phi(z) - \phi(z_0)}_{= (z-z_0)H(z),\ H \text{ holo. nonvanishing}}| - \log|z - z_0|$

$= \cancel{\log|z-z_0|} - \cancel{\log|z-z_0|} + \log|H(z)|$ is harmonic. $\square$

We'll use Green's functions to get an alternative "mapping theorem" next time.

N.B.: To clarify something in § I above: if $\Omega \cong \Omega'$, then we know the "period matrices" are the same. Up to finitely many permutations of the C_k's (basically S_n), there is only one way to get a biholomorphism to a disk minus arcs. That is to use a sum $\sum \lambda_j \omega_j$ as in the proof of the Theorem, with ${}^t\underline{\lambda} \cdot \underline{\underline{\alpha}} = (2\pi, 0, \ldots, 0)$. Since $\underline{\underline{\alpha}}$ is invertible, this determines $\underline{\lambda}$ (and hence, by determining F, the invariants $r_i, \theta_i^a, \theta_i^b$).

Conversely, if $\Omega \not\cong \Omega'$, we cannot have the same $r_i, \theta_i^a, \theta_i^b$, or Ω & Ω' would be biholomorphic to the same slit annulus, hence to each other.

Finally, that each tuple $(\{r_i\}, \{\theta_i^a\}, \{\theta_i^b\})$ occurs can be seen by considering the slit annuli with those measurements!

[So this is why the Corollary follows from the Theorem.]