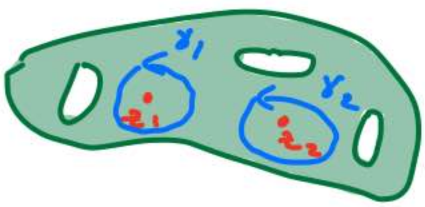
Lecture 14: Multiply connected regions (III)

I. Green's functions, cont'd.

Recall that a Green's function on a region $\Omega \subset \mathbb{C}$ with singularity at $z_0 \in \Omega$ is $g(z,z_0) \in \mathcal{H}(\Omega \setminus \{z_0\})$ s.t. $G(z) := g(z,z_0) + \log|z-z_0|$ extends to a harmonic function on a neighborhood of z_0, and $g(z,z_0) \to 0$ as $z \to \partial\Omega$. We proved uniqueness, existence (for "most" regions), that $g > 0$, and that Green's functions are compatible with biholomorphisms. We now prove two more properties:

<u>Property A</u>: $g(z_1, z_2) = g(z_2, z_1)$.

($\Rightarrow$ harmonic off $z_1 = z_2$ <u>in both variables</u>)



Proof: We use the <u>key result</u> from last term that for any two harmonic functions h_1, h_2 on a region,

$h_1 * dh_2 - h_2 * dh_1$ is a closed 1-form,

hence has periods depending only on the homology class of the path. But writing (on $\Omega \setminus \{z_1, z_2\}$) $g_i(z) = g(z, z_i)$ $(i = 1, 2)$ and $\gamma_1 + \gamma_2 \underset{\text{hom}}{\equiv} \partial\Omega$,

we have

$$\int_{\gamma_1+\gamma_2} g_1 * dg_2 - g_2 * dg_1 = \int_{\partial\Omega} g_1 * dg_2 - g_2 * dg_1 \underset{\substack{\uparrow \\ g_i \equiv 0 \text{ on } \partial\Omega}}{=} 0$$

$$\Longrightarrow \int_{\gamma_1} g_1 \underbrace{* dg_2}_{\substack{\text{harmonic} \\ \text{at } z_1}} + \int_{\gamma_2} g_1 * dg_2 = \int_{\gamma_1} g_2 * dg_1 + \int_{\gamma_2} g_2 \underbrace{* dg_1}_{\substack{\text{harmonic} \\ \text{at } z_2}}$$

Taking limits as γ_1 & γ_2 shrink, the circled terms go to 0 (as $\epsilon \log \epsilon$) and $* dg_i$ in the other two terms is reduced to $d\arg(z - z_i) =$ "$d\theta$"

$$\underset{\text{MVT}}{\Longrightarrow} \quad 2\pi\, g_1(z_2) = 2\pi\, g_2(z_1). \qquad \square$$

Property B: (Relation to ω_k)

Note the analogy to the Poisson formula

$$\frac{-1}{2\pi} \int_{C_k} * dg(z, z_0) = \omega_k(z_0)$$

for any $z_0 \in \Omega$.

Proof: $-2\pi\, \omega_k(z_0) \overset{\text{MVT}}{=} \lim_{\epsilon\to 0} \int_{C_\epsilon(z_0)} \omega_k * dg - g * d\omega_k$

$\left(C_\epsilon(z_0) \overset{\text{hom}}{\equiv} \partial\Omega \text{ on } \Omega\setminus\{z_0\} \right)$ $= \int_{\partial\Omega} \omega_k * dg - g * d\omega_k$

$\left(\begin{matrix} g \equiv 0 \text{ on } \partial\Omega \\ \omega_k \equiv \begin{cases} 0 & \text{on } \partial\Omega \setminus C_k \\ 1 & \text{on } C_k \end{cases} \end{matrix} \right)$ $= \int_{C_k} * dg .$

$\square$

The main application of all this is to obtain a different sort of canonical mapping, from Ω to $\mathbb{C}\setminus\{n \text{ vertical slits}\}$. Using the fact that locally a harmonic function is Re(holo.) hence has harmonic partial derivatives, Ahlfors obtains the

<u>Lemma</u>: If g = Green's function for Ω then

$$u_0(z) := \frac{\partial}{\partial x_0} g(z, z_0) \quad \text{is} \quad \begin{cases} \text{harmonic on } \Omega\setminus\{z_0\} \\ \text{zero on } \partial\Omega \end{cases}$$

and differs from $\mathrm{Re}\left(\frac{1}{z-z_0}\right)$ by a harmonic function.

Now, setting $A_k := \int_{C_k} * du_0$, we can use the $\{\omega_\ell\}$ to kill these periods: i.e. there is $\underline{\lambda}$ such that

$${}^t\underline{\lambda} \cdot \underline{\underline{\alpha}} = (-A_1, \dots, -A_{n-1}),$$ hence

$$u := u_0 + \lambda_1 \omega_1 + \cdots + \lambda_{n-1}\omega_{n-1}$$

such that

"supplies" the $\mathrm{Re}\left(\frac{1}{z-z_0}\right)$

"supplies" the $\mathrm{Im}\left(\frac{1}{z-z_0}\right)$

$$f = u + i\int * du \in \mathcal{Hol}(\Omega \setminus \{z_0\})$$

with a simple pole at z_0 with residue 1.

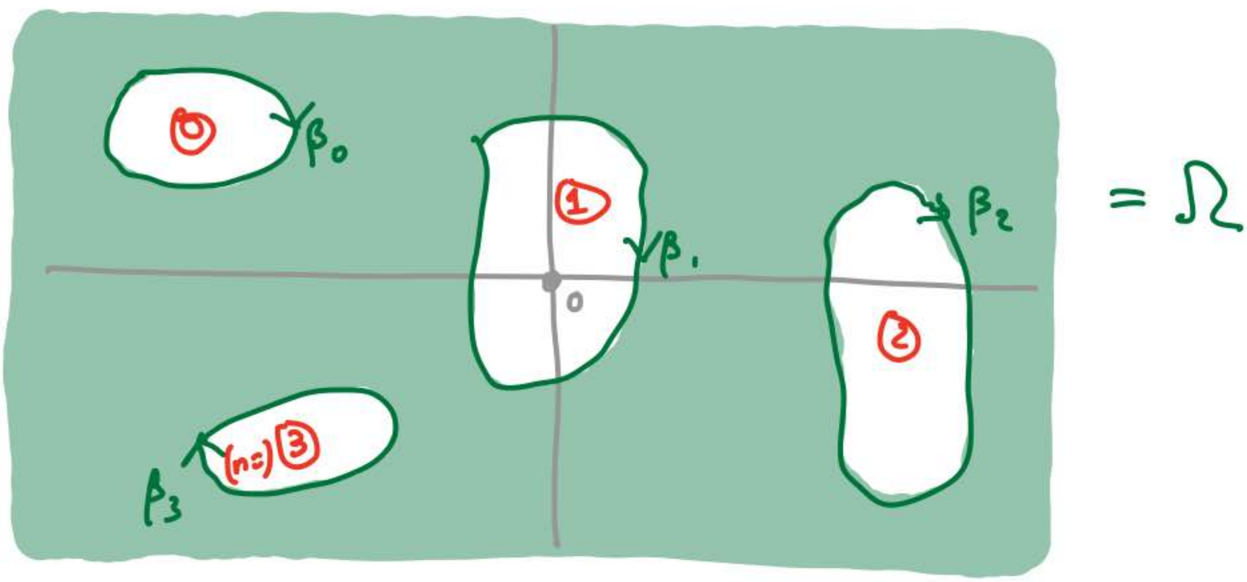
Moreover, since $u_0|_{C_k} \equiv 0$ and $\omega_\ell|_{C_k} = \delta_{\ell k}$, we have $u|_{C_k} \equiv \lambda_k$. So the C_k are mapped to slits along $\mathrm{Re}(z) = \lambda_k$. The point z_0 is sent to $\{\infty\}$. Applying the argument principle to $\int_{\partial\Omega} \frac{df}{f - w_0}$ one finds that $f(U)$ is $\hat{\mathbb{C}} \setminus {}^{\text{vertical}}_{\text{slits}}$.

<u>Exercise</u> (HW): Which annulus does the region $\mathbb{C} \setminus \{2 \text{ slits}\}$ map to? (Of course, it will depend <u>transcendentally</u> on the locations & sizes of the slits!)

II. The Schottky double

Before leaving multiply-connected regions, I want to give a beautiful relation between them and compact Riemann surfaces (complex 1-manifolds) which is apparently due to Schiffer & Spencer in the form I'll give it.

Any given n-connected domain as in Thm. 1 is biholomorphic to the complement in $\hat{\mathbb{C}}$ of n bounded simply connected regions with analytic Jordan boundary:



Write $g := n-1$, $\partial\Omega = \sum_{j=0}^{g} \beta_j$. As before, we have the harmonic measures

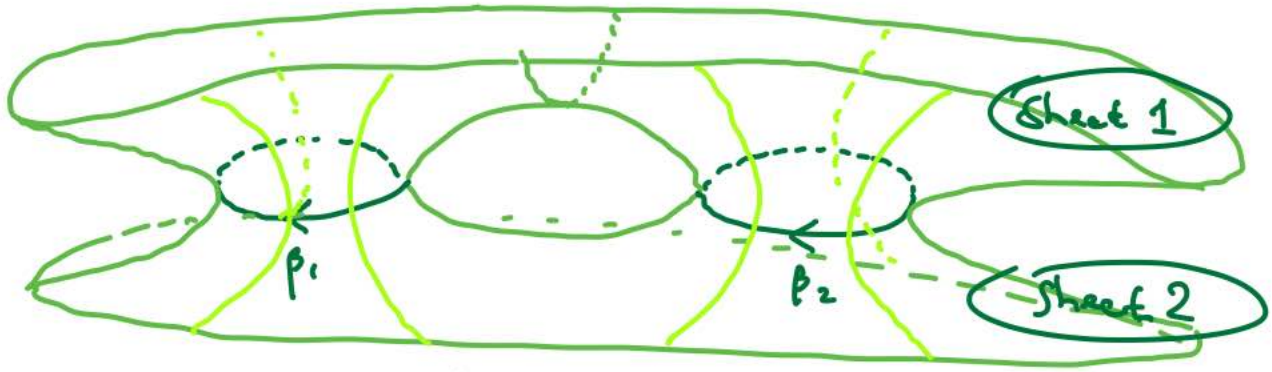
$$\omega_j(z) = \frac{1}{2\pi} \oint_{\beta_j} {*}\, d_\zeta\, g(z,\zeta) \, , \qquad j = 1, \dots, g,$$

where $\omega_j\big|_{\beta_k} \equiv \delta_{jk}$ $(k=0,\dots,g)$. Furthermore we had the periods $\Pi_{k,l} := \oint_{\beta_k} * d\omega_l$, $k,l=1,\dots,g$, which were shown to constitute a matrix of maximal rank, and which you will show is symmetric.

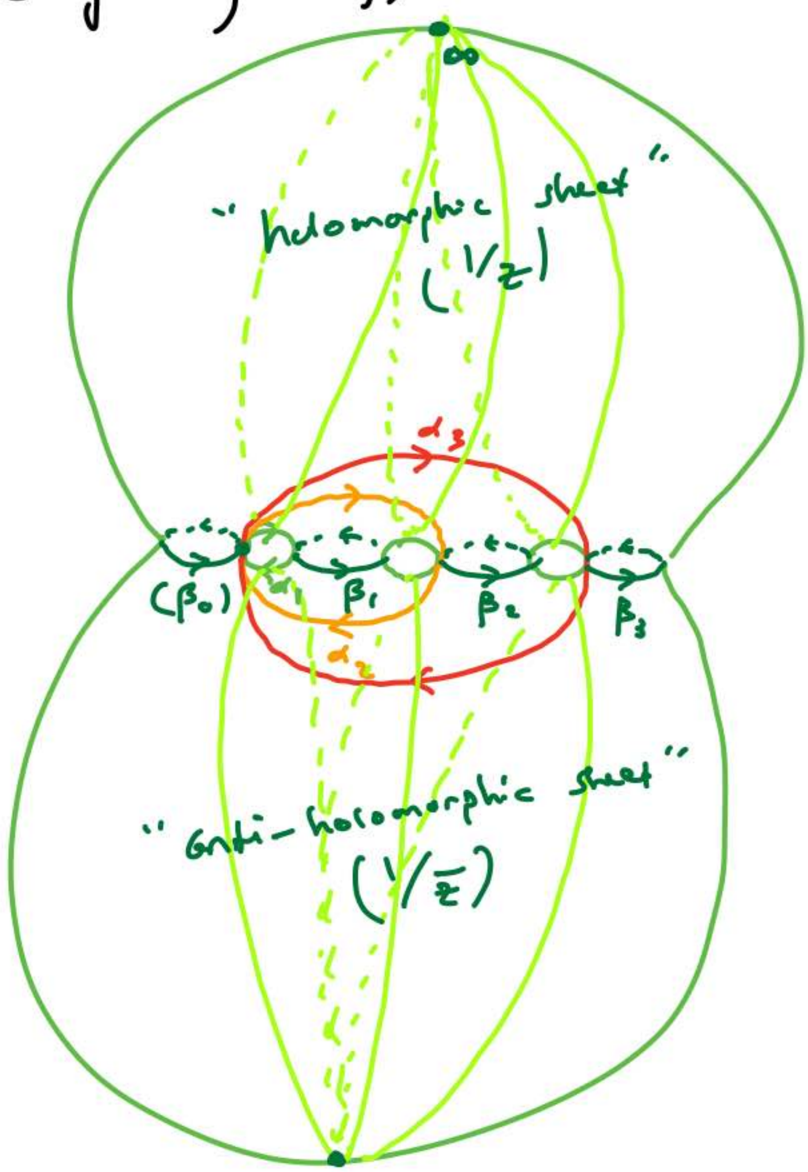
Assuming that $0 \notin \overline{\hat{\Omega}}$, $1/z$ is a holomorphic coordinate on $\hat{\Omega}$ vanishing only at $\{\infty\}$. As a set, the Schottky double is

$$\Sigma = \overbrace{\hat{\Omega} \amalg \hat{\Omega}}^{2 \text{ copies}} \amalg \partial\hat{\Omega}$$

where we $\left\{ \begin{array}{c} \text{declare } 1/z \\ \text{to be the holo.} \\ \text{coordinate} \end{array} \ \& \ \begin{array}{c} \text{declare } 1/\bar{z} \\ \text{to be the holo.} \\ \text{coordinate} \end{array} \right\}$ to get a complex analytic structure on $\Sigma \setminus \partial\hat{\Omega}$. To put an analytic structure on neighborhoods of points of $\partial\hat{\Omega}$, we declare that a meromorphic function on Σ is a pair of meromorphic functions $f, \tilde{f}$ on $\hat{\Omega}$ such that $f(z) = \tilde{f}(\bar{z})$ on the boundary. A meromorphic 1-form on Σ is a pair of meromorphic 1-forms $f(z)\,dz$ & $\tilde{f}(\bar{z})\,d\bar{z}$ which agree on the boundary. Locally, this gives you something like the picture



or more globally (say, with $n=4$)



Where the $\{\beta_j\}$ are as before, and the $\{\alpha_j\}_{j=1}^{g}$ are defined by fixing points $\xi_{i=0,1,2,\dots,g}$ on each boundary component, and going from ξ_0 to ξ_j on the holomorphic sheet then from ξ_j to ξ_0 on the antiholomorphic sheet. We have $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}\langle \alpha_1, \dots, \alpha_g; \beta_1, \dots, \beta_g \rangle$ with intersection matrix

$$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix},$$

and g is the genus of Σ.

What about holomorphic forms to integrate over these cycles? Set $W_j := \frac{1}{2}(d\omega_j + i * d\omega_j)$ on the holomorphic sheet; this has zero real part $d\omega_j$ on each β_k (since ω_j is constant there). Consequently,

$$\eta_j := \begin{cases} W_j & \text{on holo. sheet} \\ -\overline{W_j} & \text{on antiholo. sheet} \end{cases}$$

yield holomorphic 1-forms on Σ. Further, their periods are

$$\oint_{\alpha_k} \eta_\ell = \int_{\xi_0}^{\xi_k} W_\ell - \int_{\xi_k}^{\xi_0} \overline{W_\ell}$$

$$= 2\int_{\xi_0}^{\xi_k} \mathrm{Re}(W_\ell) = \int_{\xi_0}^{\xi_k} d\omega_\ell$$

$$= \delta_{k\ell}$$

and $$\oint_{\beta_k} \eta_\ell = \frac{i}{2} \oint_{\beta_k} * d\omega_\ell$$
$$= \frac{i}{2} \Pi_{k\ell}.$$

In general for Riemann surfaces, one has the result that there are always bases for H_1 and the holomorphic 1-forms yielding period matrix

$$\left(\frac{\mathbb{I}_g}{Z} \right)$$

where

- $Z = {}^t Z$ and
- $\mathrm{Im}(Z) > 0$.

By a famous result of Torelli, the matrix Z completely determines, up to change of basis for H_1, the (complex analytic) isomorphism classes of Riemann surfaces in a given family. In our case,

$$Z = \frac{i}{2} \Pi$$

and so the periods $\{\Pi_{hk}\}$ determine the conformal isomorphism class of a given Ω with fixed ordering of the boundary components. However, the proof of Torelli's theorem is well beyond the scope of this course.

Appendix: harmonic functions & differentials:

- h harmonic $\Longrightarrow$ h_x harmonic:

$$f = h + ig, \quad \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(h+ig) = \frac{1}{2}(h_x - ih_y + ig_x + g_y)$$

holo.

$$\underset{C\text{-}R}{=} h_x + ig_x = \frac{\partial f}{\partial x} \Longrightarrow h_x \text{ harmonic}$$

- Also note: for arbitrary C^1 functions F,

$$dF = \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \bar{z}}d\bar{z} = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

- h_1, h_2 harmonic $\Longrightarrow$ locally $\exists$ $f_1 = h_1 + ig_1$, $f_2 = h_2 + ig_2$

$$d(f_1 df_2) = df_1 \wedge df_2 = \frac{\partial f_1}{\partial z}dz \wedge \frac{\partial f_2}{\partial z}dz = 0$$

$(dz \wedge dz = 0)$

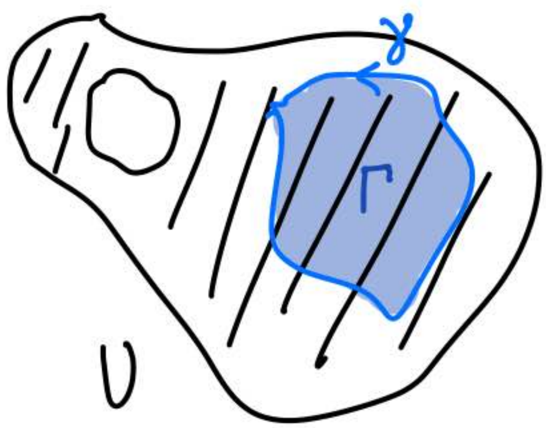
$\Rightarrow$ $f_1 df_2$ closed

$\Rightarrow$ also closed: $\operatorname{Im}(f_1\, df_2) = \operatorname{Im}\big((h_1 + i g_1)(dh_2 + i\, dg_2)\big)$

$$= h_1\, dg_2 + g_1\, dh_2 \underset{CR}{=} h_1\, dg_2 - h_2\, dg_1$$

$$= h_1 * dh_2 - h_2 * dh_1$$

- Stokes's Theorem: if $\begin{cases} \gamma = \partial\Gamma \ \underline{\underline{\text{in } U}} \\ \omega = 1\text{-form on } U \end{cases}$, then



$$\int_\gamma \omega = \int_{\partial\Gamma} \omega = \int_\Gamma d\omega.$$

So if ω is closed,

$$\int_\gamma \omega = 0 \quad \text{for } \gamma \underset{\text{hom}}{\sim} 0 \text{ in } U.$$

(e.g. $*du$, u harmonic, is closed —
it's locally the imaginary part of $d(\text{holo.})$, and $d \circ d = 0$)