Lecture 15: The Gamma function

We now turn to the part of this course which is closely related to analytic & algebraic number theory, beginning with a "differentiation lemma" whose proof will use three results you may already know (but are described in the Appendix).

↖ as Ⓐ, Ⓑ, Ⓒ

<u>Lemma:</u> Given $I \subset \mathbb{R}$ interval (possibly infinite),
$U \subset \mathbb{C}$ open, and
$f \in C^0_{\mathbb{C}}(I \times U)$ such that

(i) $\int_I f(t,z)\,dt$ is uniformly convergent on any compact $K \subset U$ $(z \in)$

i.e. given $\epsilon > 0$ $\exists\, [a,b] \subset I$ s.t. for finite intervals $J \subset I \setminus [a,b]$, $\left|\int_J f(t,z)\,dt\right| < \epsilon$ $\forall z \in K$

and

(ii) f is analytic in z (i.e. $f(t,z) \in \mathcal{H}ol(U)$ for each fixed $t \in I$),

we have

$$F(z) := \int_I f(t,z)\,dt \in \mathcal{H}ol(U)$$

and

$$F'(z) = \int_I \partial_z f(t,z)\,dt$$ (where $\partial_z f$ satisfies the same hypotheses as f itself).

Proof: Consider $I_1 \subset I_2 \subset \cdots \subset I$ $(\cup I_n = I)$,

$\bar{D} = \bar{D}(z_0, R) \subset U$.

Cauchy + (ii) $\Rightarrow$ $f(t,z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(t,\zeta)}{\zeta - z}\,d\zeta$ for $z \in D$

$$\Rightarrow \quad F(z) = \frac{1}{2\pi i}\int_I \int_{\partial D} \frac{f(t,\zeta)}{\zeta - z}\, d\zeta\, dt.$$

For $z \in \bar{D}(z_0, R/2)$, boundedness of $\left|\frac{f(t,\zeta)}{\zeta - z}\right|$ on $I_n \times \partial D$ lets us write

$$F_n(z) := \int_{I_n} f(t,z)\,dt = \frac{1}{2\pi i}\int_{I_n}\int_{\partial D} \frac{f(t,\zeta)}{\zeta - z}\, d\zeta\, dt$$

$$= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{\zeta - z} \boxed{\int_{I_n} f(t,\zeta)\,dt}\, d\zeta$$

continuous on ∂D

This is holomorphic (on $\bar{D}(z_0, R/2)$) by Ⓐ (in the Appendix).

By (i), $F_n(z) = \int_{I_n} f(t,\zeta)\,dt \xrightarrow[\text{unif. on } \bar{D}(z_0, R/2)]{} \int_I f(t,\zeta)\,dt = F(z)$,

whence $F \in \mathcal{H}ol(D(z_0, R/2))$. Since z_0 was arbitrary, and any compact $K \subset U$ can be covered by such discs, $F_n \to F$ normally on U, hence (by Ⓒ) $F_n' \to F'$ normally. Finally, by Ⓑ† we have $F_n'(z) = \int_{I_n} \partial_z f(t,z)\,dt$, whose limit is by definition $\int_I \partial_z f(t,z)\,dt$. □

Proposition: $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\,dt \in \mathcal{H}ol(U)$, where $U = \{z \mid \mathrm{Re}(z) > 0\}$.

Proof: Clearly $f(t,z) := e^{-t} t^{z-1}$ is analytic in z. We need to check uniform convergence, which we'll actually do on $K_{a,b} := \{z \mid a \le \mathrm{Re}(z) \le b\}$ where $0 < a < b < \infty$. In fact, this is clear from finiteness of $\int_0^\delta e^{-t} t^{a-1}\,dt$ and

† We know $\partial_z f(t,z)$ is continuous (for applying Ⓑ) because we can apply Ⓐ to the Cauchy integrals $f(t,z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(t,\zeta)}{\zeta - z}\,d\zeta$ to get $\partial_z f(t,z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(t,\zeta)}{(\zeta - z)^2}\,d\zeta$, which is evidently continuous in t & z because the integrand is!!

$\int_M^\infty e^{-t} t^{b-1}\,dt$ for any $a>0$ & $b<\infty$. We also see that $\Gamma'(z) = \int_0^\infty e^{-t} (\log t)\, t^{z-1}\,dt$. $\square$

Now, we would like to have an "analytic continuation" of Γ to $\mathbb{C}$ if possible; and you'll recall that the standard way to build entire functions (Γ won't be one, but $1/\Gamma$ will) was Weierstraß products!

Begin by integrating by parts:

$$\Gamma(z) = \frac{1}{z}\int_0^\infty e^{-t} t^z\,dt$$

$\begin{pmatrix} u = e^{-t}, & dv = t^{z-1}\,dt \\ du = -e^{-t}\,dt, & v = t^z/z \end{pmatrix}$

$\Longrightarrow$ $z\Gamma(z) = \Gamma(z+1)$, which together with $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$ gives $\boxed{\Gamma(n) = (n-1)!}$

$$= \frac{1}{z(z+1)}\int_0^\infty e^{-t} t^{z+1}\,dt$$

$$= \cdots$$

$$= \frac{1}{z(z+1)\cdots(z+n)}\int_0^\infty e^{-t} t^{z+n}\,dt$$

$\leftarrow$ the integral here is holomorphic in z for $\operatorname{Re}(z) > -n-1$ (!)

$$\left(= \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} \right)$$

So this gives an analytic continuation arbitrarily far to the left, but what about one expression for all of $\mathbb{C}$?

For $x \in \mathbb{R}_+$, write

$\left(\begin{array}{l} y = t/n \\ n\,dy = dt \end{array}\right)$

(*) $$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = n^x \int_0^1 (1-y)^n y^{x-1}\,dy$$

$\left(\begin{array}{l} u=(1-y)^n,\ dv = y^{x-1}\,dy \\ du = -n(1-y)^{n-1}\,dy,\ v = \frac{y^x}{x} \end{array}\right)$

$$= \frac{n^x \cdot n}{x} \int_0^1 (1-y)^{n-1} y^x\,dy$$

$$= \cdots$$

$$= \frac{n^x \cdot n!}{x(x+1)\cdots(x+n-1)} \int_0^1 y^{x+n-1}\,dy$$

$$= \frac{n^x\, n!}{x(x+1)\cdots(x+n)} =: \frac{1}{g_n(x)}.$$

Now for $t \in [0,n]$,

$$\left(1-\frac{t^2}{n^2}\right)^n = 1 - n\frac{t^2}{n^2} + \binom{n}{2}\frac{t^4}{n^4} - \binom{n}{3}\frac{t^6}{n^6} + \cdots \geq 1 - \frac{t^2}{n}$$

$$\Longrightarrow e^t\left(1-\frac{t}{n}\right)^n \geq \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n \geq 1 - \frac{t^2}{n}$$

$\left(e^{t/n} \geq 1 + \frac{t}{n}\right)$

$$\Longrightarrow \frac{t^2}{n}e^{-t} \geq e^{-t} - \left(1-\frac{t}{n}\right)^n \geq 0. \quad (**)$$

$\left(e^{-t/n} \geq 1 - \frac{t}{n}\right)$

Since $\int_0^\infty t^2 e^{-t} t^{x-1}\,dt < \infty$ $\left(\Rightarrow \int_0^n \frac{t^2}{n} e^{-t} t^{x-1}\,dt \to 0\right)$

& $\int_0^\infty e^{-t} t^{x-1}\,dt < \infty$, using (**) we

can take the limit of (*) to obtain

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt = \lim_{n\to\infty} \int_0^n e^{-t} t^{x-1}\,dt$$

$$= \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt \overset{\text{limit of } (*)}{=} \lim_{n\to\infty} \frac{1}{g_n(x)}$$

$$= \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)}$$

$$= \lim_{n\to\infty} \left\{ x \left(\prod_{k=1}^{n} \left(1+\frac{x}{k}\right) e^{-x/k} \right) e^{x\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)} \right\}^{-1}$$

$$= \left\{ x \left(\prod_{k=1}^{\infty} \left(1+\frac{x}{k}\right) e^{-x/k} \right) e^{\gamma x} \right\}^{-1}$$

where

$$\gamma := \lim_{n\to\infty} \left(1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n\right)$$

is Euler's constant.†

Theorem $\Gamma(z)^{-1} = e^{\gamma z} z \prod_{k\geq 1} \left(1+\frac{z}{k}\right) e^{-z/k}$ gives a meromorphic continuation of Γ to all of $\mathbb{C}$; moreover, $\Gamma(z)$ has simple poles at $\mathbb{Z}_{\leq 0}$ (and no zeros at all).

† The point is that the LHS converges by the Lemma and the RHS Weierstraß product is an order 1 canonical product (so convergent) $\Rightarrow \gamma < \infty$. More directly, $\gamma = 1 + \sum_{m\geq 2} \left\{ \frac{1}{m} + \log(m-1) - \log m \right\}$ with term $\frac{1}{m} + \log\left(1-\frac{1}{m}\right) = \frac{1}{m} - \frac{1}{m} + \frac{1}{2m^2} - \cdots < \frac{1}{2m^2}$ so converges!

Proof: We proved this works for $z \in \mathbb{R}_+$, and the RHS is well-defined & analytic (canonical product). But two analytic functions with the same values on a segment, agree everywhere. $\square$

Corollary 1 $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

Proof: Recall $\sin(\pi z) = \pi z \prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right)$; so

$$\Gamma(z)\Gamma(-z) = \cancel{e^{-\gamma z} e^{\gamma z}} z^{-1}(-z)^{-1} \prod_{k\geq 1}\left(1+\frac{z}{k}\right)^{-1}\left(1-\frac{z}{k}\right)^{-1} \cancel{e^{z/k} e^{-z/k}}$$

$$= -\frac{1}{z^2}\prod_{k\geq 1}\left(1-\frac{z^2}{k^2}\right)^{-1} = \frac{-\pi}{z\sin(\pi z)},$$

and $\Gamma(1-z) = -z\Gamma(-z)$. $\square$

Note in particular that this gives

$$\Gamma(1/2)\Gamma(1/2) = \frac{\pi}{\sin \pi/2} = \pi$$

$$\Longrightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}.}$$

Corollary 2 $\mathrm{Res}_{-n}(\Gamma(z)) = \dfrac{(-1)^n}{n!}$ for $n \in \mathbb{Z}_{\leq 0}$.

Proof: By Corollary 1,

$$(z+n)\Gamma(z) = \frac{\pi(z+n)}{(\sin\pi z)\Gamma(1-z)} = \frac{(-1)^n\pi(z+n)}{(\sin\pi(z+n))\,\Gamma(1-z)}$$

$$\left(\begin{aligned}\sin(\pi z+\pi n) &= \sin\pi z\cos\pi n + \cancel{\sin\pi n}\cos\pi z\\ &= (-1)^n\sin(\pi z).\end{aligned}\right)$$

and since (looking at the Weierstraß product) Γ's poles are simple,

$$\mathrm{Res}_{-n}(\Gamma(z)) = \lim_{z\to(-n)}(z+n)\Gamma(z) = \underbrace{\left(\lim_{z\to(-n)}\frac{\pi(z+n)}{\sin(\pi(z+n))}\right)}_{1}\frac{(-1)^n}{\Gamma(1+n)}$$

$$= (-1)^n/n!\,.$$

$\square$

$\boxed{\text{Corollary 3}}$ $$\prod_{j=0}^{N-1}\Gamma(z+j/N) = \frac{(2\pi)^{\frac{N-1}{2}}}{N^{Nz-1/2}}\Gamma(Nz)$$

(Gauß's multiplication formula)

Proof: LHS & RHS have no zeroes, and simple poles at $0, -\frac{1}{N}, -\frac{2}{N}, \ldots$, so the quotient is "exp" of something. Taking into account that the reciprocals of both sides are entire functions of order 1, we find that the quotient is <u>actually</u> of the form

$$A_0\cdot B_0^{\,z} \quad (!!!)$$

That is, $h(z) = AB^z = \dfrac{\prod_{j=0}^{N-1} \Gamma(z + j/N)}{\Gamma(Nz)}$

$$\Longrightarrow (B =) \frac{h(z+1)}{h(z)} = \frac{\prod_{j=0}^{N-1}(z+j/N)}{\Gamma(Nz+N)/\Gamma(Nz)} = \prod_{j=0}^{N-1} \frac{z+j/N}{Nz+j}$$

$$= N^{-N}.$$

Now $A = h(0) = \underbrace{\left(\lim_{z\to 0} \frac{\Gamma(z)}{\Gamma(Nz)}\right)}_{\text{essentially } \frac{1/z}{1/Nz}} \cdot \prod_{j=1}^{N-1} \Gamma(j/N) = N \prod_{j=1}^{N-1} \Gamma(\frac{j}{N}) > 0,$

and
$$\left(\frac{A}{N}\right)^2 = \prod_{j=1}^{N-1} \Gamma(j/N)\,\Gamma(1-j/N) \overset{\text{Cor. 1}}{=} \frac{\pi^{N-1}}{\prod_{j=1}^{N-1} \sin(\pi j/N)}$$

$$= \frac{(2\pi i)^{N-1}}{\underbrace{\prod_{j=1}^{N-1}(e^{i\pi j/N} - e^{-i\pi j/N})}_{\underbrace{\prod e^{i\pi j/N}}_{e^{i\pi(\sum j)/N} = e^{i\pi\binom{N}{2}/N} = e^{i\pi(N-1)/2} = i^{N-1}} \underbrace{\prod(1-e^{-2\pi i j/N})}_{1^{N-1}+1^{N-2}+\cdots+1 = N}}} = \frac{(2\pi \cancel{i})^{N-1}}{\cancel{i^{N-1}}\, N} = \frac{(2\pi)^{N-1}}{N}$$

(evaluation of cyclotomic polynomial at 1)

$$\Longrightarrow A = \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{N}} N = N^{1/2}(2\pi)^{\frac{N-1}{2}}. \qquad \square$$

Special case: $N = 2$ $\rightsquigarrow$ Legendre duplication formula:

$$\Gamma(z+\tfrac{1}{2})\,\Gamma(z) = \sqrt{2\pi}\;\Gamma(2z)\Big/2^{2z-1/2}$$

$$\Rightarrow \quad \Gamma(z+\tfrac{1}{2}) = \frac{\Gamma(2z)}{\Gamma(z)}\,2^{1-2z}\sqrt{\pi}\,.$$

So for example (taking $z=-1$)

$$\Gamma(-1/2) = \lim_{z\to -1}\frac{\Gamma(2z)}{\Gamma(z)}\,2^{3}\sqrt{\pi} = \cancel{\tfrac{1}{2}}\,\frac{\mathrm{Res}_{-2}\Gamma}{\mathrm{Res}_{-1}\Gamma}\,\cancel{8}^{\,4}\sqrt{\pi}$$

$$= \frac{(-1)^2/2!}{(-1)/1!}\,4\sqrt{\pi} = -2\sqrt{\pi}\,,$$

Cor. 2

and this clearly checks with our earlier computation because

$$\Gamma(\tfrac{1}{2}) = -\tfrac{1}{2}\Gamma(-\tfrac{1}{2}) = -\tfrac{1}{2}(-2\sqrt{\pi}) = \sqrt{\pi}\,.$$

Appendix: Three basic results.

(I give the proof only for (A). You should be able to supply, or look up, the ones for (B) & (C).)

Let $U \subset \mathbb{C}$ be a region, $\gamma \subset U$ a continuous path (NOT necessarily closed), and $g : \gamma \to \mathbb{C}$ a continuous function.

(A) $f(z) := \int_\gamma \frac{g(w)}{w-z}\, dw$ is holomorphic on $U \setminus \gamma$, with derivative $f'(z) = \int_\gamma \frac{g(w)}{(w-z)^2}\, dw$.

(B) Suppose $F(t,z) \in C^0([a,b] \times U)$, with $\partial_z F(t,z)$ also defined & continuous. Then $G(z) := \int_a^b F(t,z)\, dt$ is differentiable, and $G'(z) = \int_a^b \partial_z F(t,z)\, dt$.

(C) let $F_n \in \mathcal{H}ol(U)$ be a sequence converging normally (i.e. uniformly on compact subsets $K \subset U$) to F ($\in \mathcal{H}ol(U)$). Then $F_n' \to F'$ (normally).

[Sketch: use Cauchy together with (A).]

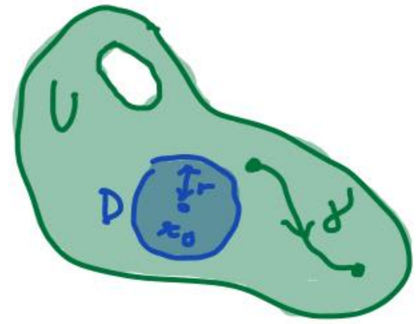
<u>Proof of (A)</u>: [<u>Note</u>: we don't have to prove $f =$ anything, hence don't have to use Cauchy $\int$ formula.]

Pick any $z \in U \setminus \gamma$, and set $r := d(z_0, \partial U \cup \gamma)$.

For all $w \in \gamma$ and $z \in D(z_0, r)$,

$$|z - z_0| < r \leq |w - z_0|$$

$$\Longrightarrow \left| \frac{z - z_0}{w - w_0} \right| < 1.$$



Thus

$$f(z) := \int_\gamma \frac{g(w)}{w - z}$$

$$= \int_\gamma \left(\sum_{k \geq 0} \frac{(z - z_0)^k g(w)}{(w - z_0)^{k+1}} \right) dw$$

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)}$$

$$= \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

$$= \frac{1}{w - z_0} \sum_{k \geq 0} \left(\frac{z - z_0}{w - w_0} \right)^k;$$

multiplying by $g(w)$, this series <u>converges uniformly</u> as functions on γ (because for $w \in \gamma$ the geometric series has $|\text{ratio}| < 1$)

$$= \sum_{k \geq 0} \int_\gamma \frac{(z - z_0)^k g(w)}{(w - z_0)^{k+1}} dw$$

$$= \sum_{k \geq 0} \left(\int_\gamma \frac{g(w)}{(w - z_0)^{k+1}} dw \right) (z - z_0)^k$$

$$=: \sum_{k \geq 0} a_k (z - z_0)^k.$$

Hence, $f(z)$ is <u>analytic</u> at z_0, with $f^{(m)}(z_0) = m!\, a_k$

$$= m! \int_\gamma \frac{g(w)}{(w - z_0)^{k+1}} dw.$$

$\square$