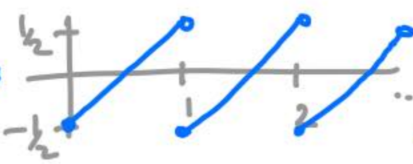
Lecture 16: Gamma and zeta

I. Stirling's formula

Lemma 1: Let $f : [0,n] \to \mathbb{R}$ be C^1.

Then $$\sum_{k=0}^{n} f(k) = \int_0^n f(t)\,dt + \frac{f(n)+f(0)}{2} + \int_0^n P_1(t) f'(t)\,dt,$$

where $P_1(t) := t - \lfloor t \rfloor - \frac{1}{2}$.

[Picture: ]

Proof: $$\int_{k-1}^{k} P_1(t) f'(t)\,dt = P_1(t) f(t)\Big|_{k-1}^{k} - \int_{k-1}^{k} f(t)\,dt$$

[$u = P_1(t)$, $dv = f'(t)dt$
$du = dt$, $v = f(t)$
(using $P_1'(t) \equiv 1$ on $(k-1, k)$)]

use $\lim_{t \to k^-} P_1(t) = 1/2$
$\lim_{t \to (k-1)^+} P_1(t) = -1/2$

$$= \frac{f(k)+f(k-1)}{2} - \int_{k-1}^{k} f(t)\,dt$$

$\sum_{k=1}^{n}$

$$\int_0^n P_1(t) f'(t)\,dt = \frac{f(0)}{2} + f(1) + \ldots + f(n-1) + \frac{f(n)}{2} - \int_0^n f(t)\,dt.$$

Now add $\frac{f(0)+f(n)}{2} + \int_0^n f(t)\,dt$ to both sides. $\square$

Lemma 2: $$\tilde{J}(z) := \int_0^\infty \frac{P_1(t)}{z+t}\,dt \in Hol(\mathbb{C} \setminus \mathbb{R}_{\le 0}).$$

[Note that this integral is <u>not</u> absolutely convergent.]

Proof: $$\int_{k-1}^{k} \frac{P_1(t)}{z+t}\,dt = \cancel{\frac{P_2(t)}{z+t}\Big|_{k-1}^{k}} \to 0 + \int_{k-1}^{k} \frac{P_2(t)}{(z+t)^2}\,dt$$

$$\left[\begin{array}{ll} u = \frac{1}{z+t}, & dv = P_1(t)\,dt \\ du = \frac{-dt}{(z+t)^2}, & v = P_2(t) \end{array}\right]$$

$P_2(t)\big|_{[0,1]} = \frac{t^2 - t}{2}$

$\sum_{k \geq 1}$

$$\int_0^\infty \frac{P_1(t)}{z+t}\,dt \overset{(*)}{=} \int_0^\infty \frac{P_2(t)}{(z+t)^2}\,dt,$$ which is absolutely

(i.e. for z in)

and uniformly convergent on any compact $K \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

So the Lemma at the beginning of Lecture 15 applies and gives the result. $\square$

<u>Remark</u>// For $z \in V_\delta := \left\{ re^{i\theta} \,\middle|\, r > 0,\ \theta \in (-\pi+\delta, \pi-\delta) \right\}$ (with any fixed $\delta > 0$), we have

$$(**) \qquad \lim_{|z| \to \infty} \int_0^\infty \frac{P_1(t)}{z+t}\,dt = 0.$$

This follows from $(*)$: given $\epsilon > 0$, if we take $R > \frac{1}{4\epsilon \sin^2\delta}$, then for $|z| = R$

$$\left| \int_0^\infty \frac{P_2(t)}{(z+t)^2}\,dt \right| \leq \frac{1}{8}\int_0^R \frac{dt}{R^2 \sin^2\delta} + \frac{1}{8}\int_R^\infty \frac{dt}{t^2 \sin^2\delta} = \frac{1}{4R\sin^2\delta} < \epsilon. \text{ //}$$

Let's apply Lemma 1 to the functions

$$\log(z+t) \quad \text{and} \quad \log(1+t):$$

$$\sum_{k=0}^{n}\left\{\log(z+k)-\log(1+k)\right\} \overset{(\dagger)}{=}$$

$$\int_0^n \log(z+t)\,dt - \int_0^n \log(1+t)\,dt$$

$$+\frac{1}{2}\left\{\log(z+n)+\log(z)-\log(1+n)-\cancelto{0}{\log(1)}\right\}$$

$$\underbrace{+\int_0^n \frac{P_1(t)}{z+t}\,dt - \int_0^n \frac{P_1(t)}{1+t}\,dt}_{\varepsilon}$$

in which the circled expression is

$$\left\{(z+t)\left[\log(z+t)-1\right]-(1+t)\left[\log(1+t)-1\right]\right\}\Big|_{t=0}^{n}$$

$$= \underbrace{z\log(z+n)}_{\substack{= z\log\left[n\left(1+\frac{z}{n}\right)\right] \\ = z\log n + z\log\left(1+\frac{z}{n}\right)}} + n\log(z+n) - z\log z + \cancel{z} - (\cancel{z}+\cancel{n}) - (n+1)\log(n+1) + (\cancel{n+1}) - \cancel{1}$$

Now moving the $z\log n$ from the RHS to LHS of $(\dagger)$, and $-\log(1+n)$ from LHS to RHS of $(\dagger)$, we have

$$\log\left(\frac{z(z+1)\cdots(z+n)}{n!\,n^{z}}\right) = z\log\left(1+\frac{z}{n}\right) - n\log\left(1+\frac{1}{n}\right) - \cancel{n\log n}$$

$$+\cancel{n\log n} + n\log\left(1+\frac{z}{n}\right) - z\log z + \cancel{\frac{1}{2}\log n}$$

$$+\frac{1}{2}\log\left(1+\frac{z}{n}\right) + \frac{1}{2}\log z - \cancel{\frac{1}{2}\log n} - \frac{1}{2}\log\left(1+\frac{1}{n}\right)$$

$$+\varepsilon$$

$$= \left(n+\frac{1}{2}\right)\log\left(1+\frac{z}{n}\right) - \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) - \left(z-\frac{1}{2}\right)\log z + z\log\left(1+\frac{z}{n}\right) + \varepsilon.$$

Taking $\lim_{n\to\infty}$, and recalling that $\lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \Gamma(z)$

(and changing sign),

$$\underset{(!)}{\log \Gamma(z)} = -z + 1 + (z - \tfrac{1}{2}) \log z - 0 + \underbrace{\int_0^\infty \frac{P_1(t)}{1+t}\,dt - \int_0^\infty \frac{P_1(t)}{z+t}\,dt}_{(\#)}$$

To evaluate (#), recalling

$$\Gamma(z)\,\Gamma(-z) = \frac{-\pi}{z \sin \pi z},$$

we have

$$\Gamma(iy)\,\underbrace{\Gamma(-iy)}_{=\overline{\Gamma(iy)}} = \frac{-\pi}{iy \sin i\pi y} = \frac{2\pi}{y\,(e^{\pi y} - e^{-\pi y})}$$

$$\Rightarrow \quad |\Gamma(iy)| = \sqrt{\frac{2\pi}{y\,(e^{\pi y} - e^{-\pi y})}}\,.$$

From (!),

$$\underbrace{1 + \int_0^\infty \frac{P_1(t)}{1+t}\,dt}_{\substack{\text{clearly real} \\ \text{\& independent of } y}} = \lim_{y\to\infty} \operatorname{Re}\left\{\log\Gamma(iy) - (iy - \tfrac{1}{2})\log(iy) + iy + \underbrace{\int_0^\infty \frac{P_1(t)}{iy+t}\,dt}_{\to 0 \text{ by Remark}}\right\}$$

$$= \lim_{y\to\infty} \left\{ \log|\Gamma(iy)| + \frac{1}{2}\log y + \frac{\pi y}{2} \right\}$$

$$= \lim_{y\to\infty} \log \sqrt{\frac{2\pi\, y\, e^{\pi y}}{y\,(e^{\pi y} - e^{-\pi y})}}$$

$$= \frac{1}{2} \log\left(\lim_{y \to \infty} \frac{2\pi}{1 - e^{-2\pi y}} \right)$$

$$= \frac{1}{2} \log 2\pi ,$$

proving the

Theorem 1 (Stirling formula)

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \log z - z + \tfrac{1}{2} \log 2\pi - E,$$

where the error term $E = \int_0^\infty \frac{P_1(t)}{z+t}\, dt$ limits to zero (as $|z| \to \infty$) in any U_δ.

Corollary $\Gamma(z) \sim z^{z - \frac{1}{2}} e^{-z} \sqrt{2\pi}$ for $|z| \to \infty$.

In particular,

$$n! = \Gamma(n+1) \sim (n+1)^{n + \frac{1}{2}} e^{-(n+1)} \sqrt{2\pi}$$

$$= n^n n^{1/2} \underbrace{\frac{\left(1 + \frac{1}{n}\right)^n}{e}}_{\to 1} \underbrace{\left(1 + \frac{1}{n}\right)^{1/2}}_{\to 1} e^{-n} \sqrt{2\pi}$$

$$\sim n^n e^{-n} \sqrt{2\pi n} .$$

II. Zeta functions

Last semester you saw a little about

- Riemann zeta $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$

(which is a priori defined for $\mathrm{Re}(s) > 1$). There are many generalizations of this function:

- Dedekind zeta function of a number field K:

$$\underbrace{\zeta_K(s)}_{\text{}} = \sum_{\substack{I \subset \mathcal{O}_K \\ \text{ideal}}} \frac{1}{(N_{K/\mathbb{Q}}(I))^s} = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \text{prime ideal}}} \frac{1}{(1 - \underbrace{N_{K/\mathbb{Q}}(\mathfrak{p})}_{\text{norm of ideal } := |\mathcal{O}_K/\mathfrak{p}|}{}^{-s})}$$

philosophy: pieces together "local" info. at primes into something "global"

by uniqueness of factorization of ideals into prime ideals

- Hurwitz zeta $\sum_{n\geq 0} \frac{1}{(n+x)^s} =: \zeta(s, x)$ $\quad x \in \mathbb{R}_{>0}$

related: Dirichlet "is" finite sum of Hurwitz with $x = \frac{m}{k}$

- Dirichlet L-function $\sum \frac{\chi(n)}{n^s}$ (χ = Dirichlet character mod k)
- L-function of a modular form $\sum \frac{a_n}{n^s}$

In arithmetic algebraic geometry, there is a huge industry devoted to checking that the "Hasse-Weil zeta function" of an algebraic variety, is in fact "modular" (= L-function of a modular form). The above types of L/zeta-functions have functional equations (conjecturally in some cases) which allow for analytic continuation and computation of residues at small/negative integers, which can carry deep information. For example, for ζ_K, the residue is given by a famous formula involving the class number & regulator of K.

We'll of course be much less ambitious here and focus on the Hurwitz & Riemann zeta functions. Start by writing

(special case: $x = 1$)

$$\Gamma(s) \underset{\substack{\uparrow \\ \mathrm{Re}(s)>0}}{=} \int_0^\infty e^{-t_0}\, t_0^s \,\frac{dt_0}{t_0} \underset{\substack{\uparrow \\ t_0=(n+x)t}}{=} \int_0^\infty e^{-(n+x)t}(n+x)^s\, t^s \,\frac{dt}{t}$$

($x > 0$ real)

$$\underset{\substack{\text{assuming} \\ \mathrm{Re}(s)>1}}{\Longrightarrow} \quad \Gamma(s)\,\zeta(s,x) = \sum_{n\geq 0} \frac{\Gamma(s)}{(n+x)^s}$$

$$= \int_0^\infty \underbrace{\sum_{n\geq 0} e^{-(n+x)t}}_{= \frac{e^{-xt}}{1-e^{-t}} =: G_x(t)} t^s \,\frac{dt}{t}$$

$$= \int_0^\infty G_x(t)\, t^s \,\frac{dt}{t}$$

which is the Mellin transform $\mathcal{M}_{G_x}(s)$ of G_x.

Now consider the path

$\gamma =$ 

and set (recalling $(-w)^s = e^{s\log(-w)}$)

$$H_x(s) := -\int_\gamma G_x(w)(-w)^s \frac{dw}{w}$$ ⟵ (clearly an entire function of s)

$$= -e^{-i\pi s}\int_\infty^\epsilon G_x(w)\, e^{s\log(w)} \frac{dw}{w}$$

$$-\int_{C_\epsilon} G_x(w)\, e^{s\log(-w)} \frac{dw}{w} - e^{i\pi s}\int_\epsilon^\infty G_x(w)\, e^{s\log(w)} \frac{dw}{w}$$

which is evidently independent of ϵ (consider the difference of two γ's). Taking the limit as $\epsilon \to 0$, and observing that $\left\| G_x(w)\, e^{s\log(-w)} \right\|_{C_\epsilon} = \left\| \frac{e^{-xw + s\log(-w)}}{1-e^{-w}} \right\|_{C_\epsilon} \leq \text{const.} \times \frac{\epsilon^s e^{\epsilon x}}{\epsilon} \to 0$ for $\underset{\sim\sim\sim}{\mathrm{Re}(s) > 1}$, we get

$$H_x(s) = -\left(e^{i\pi s} - e^{-i\pi s}\right)\int_0^\infty G_x(w)\, w^s \frac{dw}{w}$$

$$= -2i\sin(\pi s)\,\Gamma(s)\,\zeta(s,x).$$

Recalling the formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, we find that

(*******) $$\zeta(s,x) = \frac{-1}{2\pi i}\Gamma(1-s)\, H_x(s).$$ ⟵ gives analytic continuation of $\zeta(s,x)$

Now, in the definition of H_x, if we look at $s = 1-n$, $n \in \mathbb{Z}_{>0}$, then the monodromy of $(-w)^s$ goes away (leaving us only with the $\int_{C_\epsilon}$, which as before must be independent of ϵ). Hence,

$$H_x(1-n) = -\int_{C_\epsilon} (-1)^{1-n} G_x(w)\, \cancel{w^{1-n}} \frac{dw}{\cancel{w}}$$

$$= (-1)^{n-1} \int_{C_\epsilon} \frac{e^{-xw}}{e^{-w}-1} \frac{dw}{w^n}$$

$$\frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$$

where B_n = Bernoulli #s

$$= (-1)^n \sum_{\ell \geq 0} \frac{B_\ell}{\ell!} \int_{C_\epsilon} e^{(1-x)w} \frac{dw}{w^{n-\ell+1}}$$

$$= \frac{2\pi i}{n!} (-1)^n \sum_{\ell=0}^{n} \binom{n}{\ell} B_\ell (1-x)^{n-\ell}$$

and
$$\zeta(1-n, x) \underset{\text{use } (^{**}_{\ *})}{=} (-1)^{n-1} \frac{\Gamma(1-(1-n))}{n!} \sum_{\ell=0}^{n} \binom{n}{\ell} B_\ell (1-x)^{n-\ell}$$

$$= \frac{(-1)^{n-1}}{n} \sum_{\ell=0}^{n} \binom{n}{\ell} B_\ell (1-x)^{n-\ell}$$

which for $x=1$ yields the values of Riemann zeta at negative integers:

(Σ only has $\ell = n$ term)

$$\boxed{\text{Proposition}} \quad \zeta(1-n) = \zeta(1-n,1) = (-1)^{n-1} \frac{B_n}{n} = \begin{cases} -1/2, & n=1 \\ -\frac{B_n}{n}, & n \text{ even} \\ 0, & n \text{ odd} > 1 \end{cases}$$

and so in particular $\zeta(-2m) = 0$ for $m \in \mathbb{Z}_{>0}$.