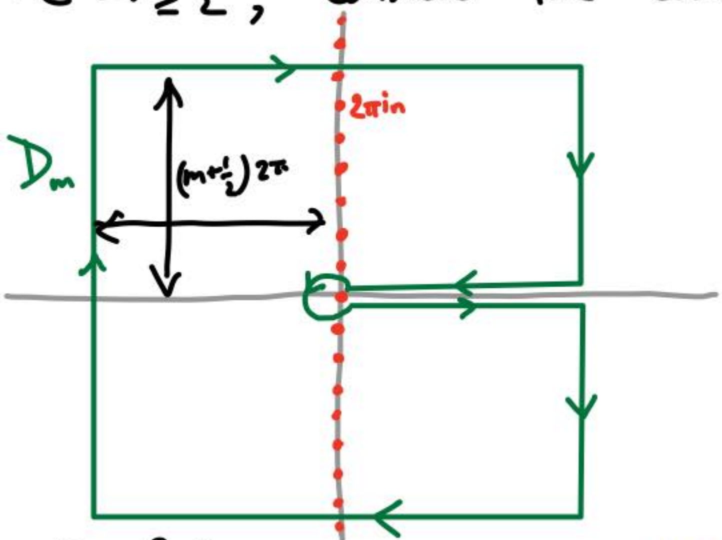
Lecture 17: More on zeta functions

I. Functional equation

The functional equation for $\zeta(s)$ allows us to understand its behavior on $\mathrm{Re}(s) < 0$ in terms of that on $\mathrm{Re}(s) > 1$, leaving only the critical strip $\mathrm{Re}(s) \in [0,1]$ as more mysterious.

For $m \in \mathbb{Z}_{\geq 2}$, consider the contour



and compute for $\mathrm{Re}(s) < 0$

$$-\int_{D_m} G_x(w)\,(-w)^s \frac{dw}{w} = -2\pi i \sum_{\lambda=-m}^{m}{}' \mathrm{Res}_{w=2\pi i\lambda}\left(\frac{(-w)^{s-1}e^{-xw}}{1-e^{-w}}\right)$$

$\ell \neq 0$

$$= -2\pi i \sum_{\ell=-m}^{m}{}' e^{-2\pi i \ell x}(2\pi|\ell|)^{s-1} e^{-\frac{i\pi s}{2}\cdot\frac{\ell}{|\ell|}} \cdot i^{\frac{\ell}{|\ell|}}$$

$$= (2\pi)^s \sum_{\ell=1}^{m} \frac{e^{-2\pi i \ell x}e^{-i\pi s/2} - e^{2\pi i \ell x}e^{i\pi s/2}}{\ell^{1-s}}.$$

$\left(\dfrac{e^{-xw}}{1-e^{-w}} \text{ has poles at } 2\pi i n\right)$

Taking the limit as $m \to \infty$, the integral over the outer square goes to 0, and so for $x \in [0,1]$ we get

$$H_x(s) = (2\pi)^s \sum_{n=1}^{\infty} \frac{-2i \sin(2\pi n x + \frac{\pi s}{2})}{n^{1-s}}.$$

For Riemann zeta ($x=1$), this gives

$$\frac{-2\pi i\, \zeta(s)}{\Gamma(1-s)} \overset{\text{by } (\ast\ast\ast) \text{ in Lec. 16}}{=} H_1(s) = -(2\pi)^s \sum_{n=1}^{\infty} \frac{e^{i\pi s/2} - e^{-i\pi s/2}}{n^{1-s}}$$

$$= -2i\,(2\pi)^s \sin\left(\tfrac{\pi s}{2}\right) \zeta(1-s)$$

$\Longrightarrow$ (for $\mathrm{Re}(s) < 0$)

$$(\dagger\dagger) \quad \zeta(s) = \zeta(1-s) \times \Gamma(1-s) \frac{\sin(\pi s/2)}{\pi} (2\pi)^s.$$

But since we know (by entireness of $H_1(s)$) that ζ is "entire meromorphic", it follows that $(\dagger\dagger)$ holds on $\mathbb{C}$.

$\boxed{\text{Theorem 2}}$ (functional equation for ξ) Let $\xi(s) := s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Then ξ is an entire function satisfying the functional equation

$$\xi(s) = \xi(1-s).$$

Proof: Recall that

$$\Gamma(u)\,\Gamma(u+\tfrac{1}{2}) = \left(2\sqrt{\pi}/2^{2u}\right)\Gamma(2u)$$

and

$$\Gamma(u)\,\Gamma(1-u) = \pi/\sin(\pi u).$$

Together with (††) these imply that $\xi(s)$ is invariant under $s \mapsto 1-s$. [Exercise]

Furthermore, recalling $\underbrace{H_1(s)}_{\text{entire}} = -2i\underbrace{(\sin(\pi s))\Gamma(s)}_{\text{zeros only at } \mathbb{Z}_{>0}}\zeta(s)$,

we see that $\zeta(s)$ can have poles only at $\mathbb{Z}_{>0}$. But $\sum \frac{1}{n^s}$ is holomorphic for $\mathrm{Re}(s) > 1$, so the only possible pole is at $s = 1$. From (††), we see this is a simple pole, so that $(s-1)\zeta(s)$ is actually entire. Since ξ is therefore holomorphic on $\mathrm{Re}(s) > 0$, it must (by the $s \mapsto 1-s$ invariance) be so everywhere. $\square$

II. Zeroes and poles

Recall that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

which evidently has no poles, and no zeroes apart from those of the individual factors. So

$\underline{\Gamma(z)\text{ has no zeroes, and no poles}}$
$\underline{\text{apart from a simple pole at each } n \in \mathbb{Z}_{\leq 0}.}$

For the Hurwitz zeta function $\zeta(s,x)$, we had

$$\zeta(s,x) = -\frac{1}{2\pi i} \Gamma(1-s) H_x(s),$$

with $H_x(s)$ entire. Hence, the only possible poles are at $m \in \mathbb{Z}_{>0}$ and are (at worst) simple poles. But for $m \in \mathbb{Z}_{\geq 2}$, we know $\sum_{n \geq 0} \frac{1}{(n+x)^m} < \infty$. So the only pole of $\zeta(s,x)$ possible (in s) is a simple pole at $s = 1$. To see that it is in fact a pole, the easiest approach at this point is to compute†

$$H_x(1) = \int_{C_\epsilon} \frac{e^{-wx}}{1-e^{-w}} \cancel{w^s} \frac{dw}{\cancel{w}} = 2\pi i \, \mathrm{Res}_0\left(\frac{e^{-wx}}{1-e^{-w}}\right)$$

$$= 2\pi i.$$

For the Riemann zeta, I want to give a more direct proof:

Lemma 1: The function $\zeta(s) - \frac{1}{s-1}$ extends to a holomorphic function on $\mathrm{Re}(s) > 0$. (So using the functional equation $\zeta(s) = (2\pi)^s \Gamma(1-s) \frac{\sin\left(\frac{\pi s}{2}\right)}{\pi} \zeta(1-s)$ to extend ζ to $\mathrm{Re}(s) < 1$, we see that it is pole-free in that region.)

Proof: For $\mathrm{Re}(s) > 1$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n \geq 1} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s}\, dx$$

$$= \sum_{n \geq 1} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$$

† also note that $H_x(m) = 0$ for $m \geq 2$

where $\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leq \left\| \frac{1}{n^s} - \frac{1}{x^s} \right\|_{[n,n+1]}$

take d/dx

(since function is 0 at $x=n$) $\rightsquigarrow \leq \left\| \frac{s}{x^{s+1}} \right\|_{[n,n+1]}$

$$\leq \frac{|s|}{n^{\mathrm{Re}(s+1)}}.$$

So the sum above has absolute value bounded by

$$|s| \sum_{n \geq 1} \frac{1}{n^{\mathrm{Re}(s)+1}}$$

which converges for $\mathrm{Re}(s) > 0$. $\square$

What about the zeroes of $\zeta(s)$? This is where the product development comes in.

Theorem 1 For $\mathrm{Re}(s) > 1$, we have the "Euler product" expansion $\zeta(s) = \prod_{p>1 \text{ prime}} (1-p^{-s})^{-1}$.

Proof: Recall that for a sequence $\{a_n\} \subset \mathbb{C}$

$$\prod (1-a_n) \text{ AC \& UC} \iff \sum |a_n| \text{ UC}.$$

So we need $\sum p^{-s}$ UC, which is clear since

$$\sum_{n\ge 1} n^{-s} \le \sum_{n\ge 1} n^{-(1+\epsilon)} \left(\le 1+\int_1^\infty \frac{dx}{x^{1+\epsilon}} = 1+\frac{1}{\epsilon}\right)$$

for $\mathrm{Re}(s) \ge 1+\epsilon$ (for any $\epsilon > 0$).

Now informally

$$\frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \cdots =$$

$$(1+2^{-s}+4^{-s}+8^{-s}+\cdots)(1+3^{-s}+9^{-s}+\cdots)(1+5^{-s}+25^{-s}+\cdots)\cdots =$$

$$1+2^{-s}+3^{-s}+4^{-s}+5^{-s}+6^{-s}+\cdots$$

which works by uniqueness of factorization into primes (so that each n^{-s} occurs with multiplicity one in this product).

More formally

$$\zeta(s)(1-2^{-s}) = \sum_{n\ge 1} n^{-s} - \sum_{n\ge 1} (2n)^{-s} = \sum_{m \text{ odd}} m^{-s}$$

$$\zeta(s)(1-2^{-s})(1-3^{-s}) = \sum_{m \text{ odd}} m^{-s} - \sum_{m \text{ odd}} (3m)^{-s} = \sum_{2,3\nmid k} k^{-s} \le 1+\sum_{n\ge 5} n^{-s}$$

$$\zeta(s)(1-\overset{p_1}{2}{}^{-s})\cdots(1-p_N^{-s}) = \sum_{2,\dots,p_N \nmid k} k^{-s} \le 1+\sum_{n\ge p_{N+1}} n^{-s} \xrightarrow[N\to\infty]{} 1$$

(using the infinitude of primes). $\square$

Now because of the Euler product, which has no term zero for $\mathrm{Re}(s) > 1$, ζ has <u>no zeroes</u> there.

For $\mathrm{Re}(s) < 0$, use the functional equation

$$\zeta(s) = (2\pi)^s \underbrace{\Gamma(1-s)}_{\substack{\text{no zeroes}\\ \text{or poles if}\\ \mathrm{Re}(s)<0}} \underbrace{\frac{\sin(\pi s/2)}{\pi}}_{\substack{\text{zeroes at}\\ s \in 2\mathbb{Z}}} \underbrace{\zeta(1-s)}_{\substack{\text{we just showed}\\ \text{that this has}\\ \text{no zeroes}\\ \text{for } \mathrm{Re}(s)<0}}$$

to deduce that <u>$\zeta(s)$ has simple zeroes at negative even integers.</u> So we arrive at the

Proposition <u>The zeroes of $\zeta(s)$ are at $s \in 2\mathbb{Z}_{<0}$ and in the critical strip $0 \le \mathrm{Re}(s) \le 1$. It has one (simple) pole, with residue 1, at $s = 1$.</u>

\$1,000,000 Conjecture (Riemann Hypothesis)

<u>The zeroes of $\zeta(s)$ are at $2\mathbb{Z}_{<0}$ and on the line $\mathrm{Re}(s) = \frac{1}{2}$.</u>

<u>Remark</u> // By the definition of $\zeta(s)$, the fact that $\overline{n^s} = \overline{e^{s\log n}} = \overline{\sum \frac{(s\log n)^k}{k!}} = \sum \frac{(\bar{s}\log n)^k}{k!} = n^{\bar{s}}$,

and the Lemma, we see that $\zeta(\bar{s}) = \overline{\zeta(s)}$ for $\mathrm{Re}(s) > 0$; by the functional equation, this is clear for all s.

Upshot: <u>if ζ has a zero of order n at α, then it has a zero of order n at $\bar{\alpha}$.</u>

(Also recall that there isn't one at 0, where $\zeta(0) = -\frac{1}{2}$, or 1, where ζ has a pole.) //

Since the ζ-function is connected to primes by the Euler product expansion, one might imagine that the Riemann Hypothesis would have profound implications for their distribution. For example, one has the following result of Cramér: RH $\Rightarrow$ the gap between prime p and the next prime is bounded by a constant times $\sqrt{p}\log(p)$. (We'll see another consequence when we study the Prime Number Theorem.)