Lecture 21: Lattices in $\mathbb{C}$

I. Discrete subgroups & fundamental parallelograms

Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ ($\Leftrightarrow \frac{\omega_2}{\omega_1} \notin \mathbb{R}$). The <u>lattice</u> generated by ω_1, ω_2 is the abelian group

$$\Lambda := \mathbb{Z}\langle \omega_1, \omega_2 \rangle = \{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z} \}.$$

The set of equivalence classes of complex numbers modulo Λ is the quotient group

$$\mathbb{C}/\Lambda \overset{\text{as}}{\underset{\text{set}}{:=}} \{ \underbrace{z + \Lambda}_{\text{think of as coset}} \mid z \in \mathbb{C} \}$$

A <u>fundamental parallelogram</u> for Λ is (non-uniquely)

$$\widetilde{\mathcal{F}}_\Lambda := \{ \alpha + t_1\omega_1 + t_2\omega_2 \mid t_1, t_2 \in [0,1) \}. \quad (\alpha \in \mathbb{C} \text{ fixed})$$

<u>Lemma 1</u>: (a) For any $z \in \mathbb{C}$, there exists a unique $z_0 \in \widetilde{\mathcal{F}}_\Lambda$ such that $z \underset{\Lambda}{\equiv} z_0$. ($\Leftrightarrow z + \Lambda = z_0 + \Lambda$)

(b) $|\widetilde{\mathcal{F}}_\Lambda \cap \Lambda| = 1$.

Proof: (b) is just (a) applied to $z \in \Lambda$.

(a) By linear algebra, we have uniquely

$$z - \alpha = a\omega_1 + b\omega_2, \qquad a, b \in \mathbb{R}.$$

Taking $z_0 := z - \lfloor a \rfloor \omega_1 - \lfloor b \rfloor \omega_2 \ (\underset{\Lambda}{\equiv} z)$,

$$z_0 - \alpha = (z-\alpha) - (\lfloor a\rfloor \omega_1 + \lfloor b \rfloor \omega_2) = \underbrace{(a-\lfloor a\rfloor)}_{\in[0,1)}\omega_1 + \underbrace{(b-\lfloor b\rfloor)}_{\in[0,1)}\omega_2$$

If z_0' is another such, then

$$\Lambda \ni z_0 - z_0' = \overbrace{(t_1 - t_1')}^{\in(-1,1)}\omega_1 + \overbrace{(t_2 - t_2')}^{\in(-1,1)}\omega_2$$

$$\Longrightarrow t_1 - t_1',\ t_2 - t_2' \in \mathbb{Z}$$

$$\Longrightarrow z_0 = z_0'. \qquad \square$$

<u>Corollary</u>: $\mathcal{F}_\Lambda$ is a <u>fundamental domain</u> for Λ, i.e.

$$\mathbb{C}/\Lambda \underset{\text{set}}{\overset{\text{as}}{\cong}} \mathcal{F}_\Lambda .$$

Now let $A \subset \mathbb{C}$ be a (sub) abelian group.

A is <u>discrete</u> $\overset{\text{defn.}}{\Longleftrightarrow}$ each $a \in A$ has a neighborhood $\mathcal{N} \subset \mathbb{C}$ s.t. $\mathcal{N} \cap A = \{a\}$.

$\boxed{\text{Proposition}}$ $\Longleftrightarrow A = \{0\}$, $\mathbb{Z}\langle \omega \rangle$, or $\mathbb{Z}\langle \omega_1, \omega_2 \rangle$ (where $\omega_2/\omega_1 \notin \mathbb{R}$).

Proof: Assume $A \neq \{0\}$. Some $\bar{D}_r \cap A \neq \{0\}$; but also $|\bar{D}_r \cap A| < \infty$ by discreteness of A & compactness of $\bar{D}_r$.

Let $\omega_1 \in \bar{D}_r \cap A \setminus \{0\}$ be an element with minimal modulus.

Then $A \supset \mathbb{Z}\langle \omega_1 \rangle$.

Suppose $A \neq \mathbb{Z}\langle \omega_1 \rangle$, and let $\omega_2 \in A \setminus \mathbb{Z}\langle \omega_1 \rangle$ have minimal modulus (again using discreteness). Then $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$ [otherwise $\frac{\omega_2}{\omega_1} \in \mathbb{R} \setminus \mathbb{Z}$, so $|\omega_2 - \lfloor \frac{\omega_2}{\omega_1} \rfloor \omega_1| \in (0, |\omega_1|)$ $\times\!\!\!\times$].

By linear algebra, $\mathbb{R}\langle \omega_1, \omega_2\rangle = \mathbb{C}$. If $\omega \in A$ then taking $\alpha = -\frac{1}{2}(\omega_1+\omega_2)$, we let $\mathcal{F}_\Lambda$ denote the fundamental domain for $\Lambda := \mathbb{Z}\langle \omega_1, \omega_2\rangle$. There exists a unique $\omega_0 \in \mathcal{F}_\Lambda$ with $\omega_0 \underset{\Lambda}{\equiv} \omega$, and clearly $\omega_0 \in A$ ($\Rightarrow \omega_0 \neq \alpha$). But then

$$|\omega_0| < \tfrac{1}{2}|\omega_1| + \tfrac{1}{2}|\omega_2| \leq |\omega_2| \underset{\text{choice of } \omega_2}{\Longrightarrow} \omega_0 \in \mathbb{Z}\langle \omega_1\rangle$$

$$\Longrightarrow \omega \in \Lambda. \qquad \square$$

II. Bases, period ratios, and $\mathfrak{h}/SL_2(\mathbb{Z})$

Given Λ, one can choose a basis $\{\omega_1, \omega_2\}$ as in the Proposition, but this choice is neither unique nor a priori desirable. Since $\Lambda \cong \mathbb{Z}^2$ as abelian groups, we may analyze choices of basis in the latter. Consider $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ and v_1, v_2 bases of $\mathbb{Z}^2$; then the transformation $\mathbb{Z}^2 \to \mathbb{Z}^2$ sending $e_i \mapsto v_i$ ($i=1,2$) is described by the integral matrix $\begin{pmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{pmatrix}$.

Since $\{v_i\}$ is a basis, the map is injective & surjective hence invertible $\Rightarrow$ M^{-1} <u>integral</u> $\Rightarrow M \in GL_2(\mathbb{Z})$ ($\Rightarrow \det M = \pm 1$).

This gives

<u>Lemma 2</u>: Any 2 bases for Λ are related by

$$\begin{aligned} \omega_2 &\longmapsto a\omega_2 + b\omega_1 \\ \omega_1 &\longmapsto c\omega_2 + d\omega_1 \end{aligned}, \qquad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}).$$

Important Remark// Under this transformation, the period ratio

$$\tau := \frac{\omega_2}{\omega_1} \longmapsto \frac{a\tau+b}{c\tau+d} =: \gamma\langle\tau\rangle$$

and it suffices to consider $\pm\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PGL_2(\mathbb{Z})$ to see the effect on τ. We'll usually stick to changes of basis preserving "orientation", i.e. those (with det $= +1$) in $\Gamma := SL_2(\mathbb{Z})$.

Note that since $\mathfrak{h}$ is closed under $SL_2(\mathbb{R})$, $\tau \in \mathfrak{h}$ & $\gamma \in \Gamma$ $\Rightarrow \gamma\langle\tau\rangle \in \mathfrak{h}$. Any 2 bases with period ratios in $\mathfrak{h}$ are related by $\gamma \in \Gamma$. //

Occasionally, a transformation of basis may be described via "complex multiplication":

$\boxed{\text{Definition}}$ Λ has CM $\iff \exists\, \gamma \in \mathbb{C}\setminus\mathbb{Z}$ s.t. $\gamma\Lambda = \Lambda$.

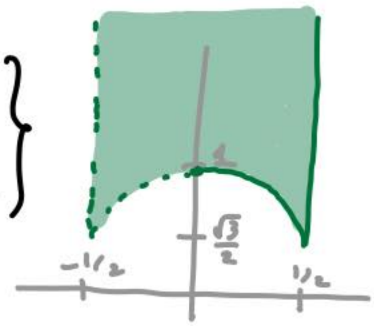
Example// $\Lambda = \mathbb{Z}\langle 1, i\rangle$ mult. by i

$\Lambda = \mathbb{Z}\langle \zeta_6, 1\rangle$ mult. by ζ_6

(Notice that CM does not affect the period ratio τ — it just changes the basis.) //

Remark// The definition can be weakened to say $\gamma: \Lambda \to \Lambda$ gives an endomorphism (we require automorphism). That definition isn't relevant here but in that case any $\mathbb{Z}\langle 1, \tau\rangle$ with τ satisfying a quadratic equation $\tau^2 = a\tau + b$, $a, b \in \mathbb{Z}$, will work. //

Set $\mathcal{F}_\Gamma := \left\{ \tau \in \mathfrak{h} \;\middle|\; \begin{array}{l} \mathrm{Re}(\tau)\in(-\frac{1}{2},\frac{1}{2}] \\ |\tau|\ge 1 \\ \mathrm{Re}(\tau)\ge 0 \text{ if } |\tau|=1 \end{array}\right\}$



and let $\Lambda \subset \mathbb{C}$ be any lattice.

$\boxed{\text{Theorem 1}}$ Λ has a unique period ratio τ in $\mathcal{F}_\Gamma$. If $\tau \neq i, \zeta_6$ then there are two choices of basis (say, $\{\omega_1,\omega_2\}$ and $\{-\omega_1,-\omega_2\}$) yielding this τ. If $\tau = i$ resp. ζ_6, then there are 4 resp. 6 such choices.

$\boxed{\text{Corollary A}}$ $\tau = i$ or ζ_6 are the only possibilities (in $\mathcal{F}_\Gamma$) for Λ to have CM!

Proof of Thm. 1:

$\boxed{\text{STEP 1}}$ ($\exists$ of τ)

Choose ω_1, ω_2 as in the proof of the Proposition (note $\tau \notin \mathbb{R}$). By taking $\omega_2 \mapsto \pm\omega_2$ we get $\tau \in \mathfrak{h}$. Now that proof $\Rightarrow$

$$\begin{cases} |\omega_1| \le |\omega_2| & (\text{choice of } \omega_1) \\ |\omega_2| \le |\omega_1 \pm \omega_2| & (\text{choice of } \omega_2) \end{cases} \Longrightarrow$$

$$\begin{cases} 1 \le |\tau| \\ |\tau| \le |1 \pm \tau| \end{cases}$$

$\Rightarrow x^2 + y^2 \le (x \pm 1)^2 + y^2$
$\Rightarrow |x| \le |x \pm 1| \Rightarrow -x-1 \le x \le -x+1$
$\Rightarrow -1 \le 2x \le 1 \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$, replace ω_2 by $\omega_1+\omega_2$ $\rightsquigarrow$ $\tau \mapsto \tau+1$.

If $|\tau|=1$ and $\mathrm{Re}(\tau)<0$, replace (ω_1,ω_2) by $(-\omega_2,\omega_1)$

$\rightsquigarrow$ $\tau\,(=e^{i\theta}) \mapsto -\frac{1}{\tau}\,(=e^{i(\pi-\theta)})$.

$\boxed{\text{STEP 2}}$ (! of τ)

We must show that if τ,τ' (= period ratios for Λ) $\in \mathfrak{h}$ lie in $\bar{\mathcal{F}}_{\Gamma}$, then $\tau\,(=x+iy) = \tau'$. They are related by $SL_2(\mathbb{Z})$:

$$\tau' = \frac{a\tau+b}{c\tau+d},\quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma \implies \mathrm{Im}(\tau') = \mathrm{Im}\,\frac{(a\tau+b)(c\bar{\tau}+d)}{|c\tau+d|^2}$$

$$= \mathrm{Im}\,\frac{iy\overbrace{(ad-bc)}^{=1}+\cdots}{|c\tau+d|^2}$$

$$= \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$$

(we may assume $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau)$

$$\implies 1 \geq |c\tau+d|^2 = |(cx+d)+icy|^2 = (cx+d)^2+y^2c^2$$

$$\geqslant (cx+d)^2 + \frac{3c^2}{4}$$

$\implies |c| \leq 1$.

If $c=\pm 1$, $1 \geq (d\pm x)^2 + \frac{3}{4} \implies \frac{1}{4} \geq (d\pm x)^2$

$\implies |d\pm x| \leq \frac{1}{2} \implies d=0$.

If $c=0$, then $|d|\leq 1$

$\implies (c,d) = (\pm 1, 0)$ or $(0,\pm 1) \implies \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \pm\begin{pmatrix} a & -1 \\ 1 & 0\end{pmatrix}$ or $\pm\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$

$\implies \tau' = \overset{(i)}{a-\frac{1}{\tau}}$ or $\overset{(ii)}{\tau+b}$ (with $\tau,\tau' \in \bar{\mathcal{F}}_{\Gamma}$).

In case (ii), $b=0$ (and $\tau=\tau'$).

In case (i) with $a \neq 0$, $\left|\frac{1}{\tau}\right| = |a - \tau'| \geq 1$ (look at distance between $\mathcal{F}_\Gamma$ and $[1,\infty) \cup (-\infty,-1]$)

$\Rightarrow |\tau| \leq 1 \underset{\tau \in \mathcal{F}_\Gamma}{\Rightarrow} |\tau| = 1 \Rightarrow \tau' = a - \bar{\tau}$

$(\mathcal{F}_\Gamma \Rightarrow)$ $= a + e^{i(\pi - \theta)}$, impossible unless $\tau = \tau' = \zeta_6$ & $a = 1$.

But if $a = 0$, $\tau' = -\frac{1}{\tau}$ and τ can't both be in $\mathcal{F}_\Gamma$ either, unless both $= i$.

STEP 3 (choices of bases yielding τ)

This is now easy: the transformations between such bases are precisely the matrices $\gamma \in SL_2(\mathbb{Z})$ fixing τ. We found in Step 2 that unless $\tau = i$ or ζ_6, case (ii) isn't viable; and in case (i) $b = 0 \Rightarrow \gamma = \pm \mathrm{id}$. If $\tau = i$ the possible matrices are powers of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and for $\tau = \zeta_6$ powers of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. $\square$

Corollary B $\mathcal{F}_\Gamma$ is a fundamental domain for $SL_2(\mathbb{Z})$, i.e.

$$\Gamma \backslash \mathfrak{h} \underset{\text{sets}}{\overset{\text{as}}{\cong}} \mathcal{F}_\Gamma .$$