Lecture 23: Elliptic functions & elliptic curves

I. Weierstraß σ- & ζ-functions

Let $\Lambda := \mathbb{Z}\langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$, $\operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$.

In Lecture 22 we proved that

$$\sigma(z) := z \prod_{\omega\in\Lambda}{}' \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

omit $\omega = 0$

defines an entire function with simple zeros on Λ (only).

Let $\omega_0 \in \Lambda$; then

$$\frac{\sigma(z+\omega_0)}{\sigma(z)} = \frac{(z+\omega_0) \prod_{\omega\in\Lambda}' \left(1 - \frac{z+\omega_0}{\omega}\right) e^{\frac{z+\omega_0}{\omega} + \frac{1}{2}\left(\frac{z+\omega_0}{\omega}\right)^2}}{z \prod_{\omega\in\Lambda}' \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}},$$

and logarithmically differentiating yields

$$\frac{d}{dz}\log\left(\frac{\sigma(z+\omega_0)}{\sigma(z)}\right) = \frac{1}{z+\omega_0} - \frac{1}{z} + \underbrace{\sum{}' \left\{\frac{1}{z+\omega_0-\omega} - \frac{1}{z-\omega} + \frac{\omega_0}{\omega^2}\right\}}_{\text{Almost, but not yet, a collapsing sum. Need to differentiate once more...}}$$

$$\left(\frac{d}{dz}\right)^2\log\left(\frac{\sigma(z+\omega_0)}{\sigma(z)}\right) = \frac{-1}{(z+\omega_0)^2} + \frac{1}{z^2} + \underbrace{\sum{}' \left\{\frac{-1}{(z+\omega_0-\omega)^2} + \frac{1}{(z-\omega)^2}\right\}}_{\text{can now collapse. Note that } \sum' \text{ contains no } \frac{-1}{(z+\omega_0)^2} \text{ \& no } \frac{1}{z^2}.}$$

$$= \frac{-1}{(z+\omega_0)^2} + \frac{1}{z^2} + \frac{-1}{z^2} + \frac{1}{(z+\omega_0)^2}$$

$$= 0.$$

So $\log\left(\frac{\sigma(z+\omega_0)}{\sigma(z)}\right) \underset{\text{def.}}{=:} \eta(\omega_0)\, z + \zeta(\omega_0)$ (is linear).

Now clearly

$$\sigma(-z) = (-z)\prod_{\omega}{}' \left(1+\frac{z}{\omega}\right) e^{-\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

$$\underset{(\omega \mapsto -\omega)}{=} -\sigma(z) \qquad \text{(is odd)}$$

From the above, we have

$$\sigma(z+\omega_0) = \sigma(z)\, e^{\eta(\omega_0) z + \zeta(\omega_0)};$$

and setting $z = -\frac{\omega_0}{2}$ gives, assuming $\frac{\omega_0}{2} \notin \Lambda$,

$$\sigma\left(\frac{\omega_0}{2}\right) = \sigma\left(\frac{-\omega_0}{2}\right) e^{-\eta(\omega_0)\frac{\omega_0}{2} + \zeta(\omega_0)}$$

$$= -\sigma\left(\frac{\omega_0}{2}\right) e^{-\eta(\omega_0)\frac{\omega_0}{2} + \zeta(\omega_0)}$$

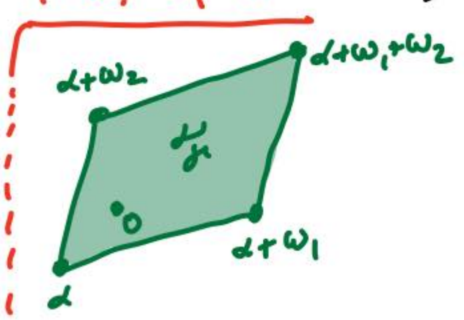
$\Rightarrow \zeta(\omega_0) = \pi i + \eta(\omega_0)\frac{\omega_0}{2}$. This proves the 1st part of

Theorem 1 (a) $\sigma(z+\omega_i) = -\sigma(z)\, e^{\eta_i \cdot \left(z + \frac{\omega_i}{2}\right)}$ $(i=1,2)$

where $\eta_i := \eta(\omega_i)$.

(b) $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ [Legendre relation]

Proof of (b): Define the Weierstraß ζ-function



$$\zeta(z) := \frac{d}{dz}\log(\sigma(z))$$

$$= \frac{1}{z} + \sum_{\omega\in\Lambda}{}' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

Since ζ has simple poles on Λ and nowhere else, we have

$$
\begin{aligned}
2\pi i &= 2\pi i \operatorname{Res}_0 \zeta = 2\pi i \sum_{p \in \mathcal{F}} \operatorname{Res}_p \zeta \\
&= \int_{\partial \mathcal{F}} \zeta(z)\, dz = \int_{\alpha}^{\alpha+\omega_1} \underbrace{\big(\zeta(z) - \zeta(z+\omega_2)\big)}_{-\eta_2} dz + \int_{\alpha}^{\alpha+\omega_2} \underbrace{\big(\zeta(z+\omega_1) - \zeta(z)\big)}_{\eta_1} dz \\
&= -\eta_2 \omega_1 + \eta_1 \omega_2 .
\end{aligned}
$$

by logarithmically differentiating part (a). $\square$

II. Weierstraß $\wp$-function

Set $\wp(z) := -\zeta'(z)$

$$
= \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) .
$$

Properties

① $\wp$ is even [Proof: σ odd $\Rightarrow \zeta = \frac{\sigma'}{\sigma}$ (even) odd $\Rightarrow \zeta'$ even.]

$$\Rightarrow \wp'(z) = -\frac{2}{z^3} - \sum{}' \frac{1}{(z-\omega)^3} \quad \text{is odd.}$$

② $\wp$ is elliptic (biperiodic) [Proof: $\sigma(z+\omega_i) = -\sigma(z) e^{\eta_i (z + \frac{\omega_i}{2})} \xRightarrow{\frac{d}{dz}\log(\cdot)}$
$\zeta(z+\omega_i) = \zeta(z) + \eta_i \xRightarrow{d/dz}$
$\wp(z+\omega_i) = \wp(z)$, $i=1,2$.]

$\Longrightarrow \wp'$ is also elliptic.

③ $\wp$ (resp. $\wp'$) is of mapping degree 2 (resp. 3) as a map from $\mathbb{C}/\Lambda$ to $\mathbb{P}^1$.

[Proof: let $z_0 \in \mathcal{F}$, consider $\wp(z) - \wp(z_0)$. We want to show that it has 2 0's in $\mathcal{F}$, counting multiplicity. It has one pole, with mult. (-2), and $\sum m_i = 0$ (Lect. 22) then implies the result. $\wp'$ is treated in the same way.]

④ $\wp$ (resp. $\wp'$) has principal part $1/z^2$ (resp. $-2/z^3$) at 0.

[Proof: obvious from property ①, & differentiating.]

Though I haven't listed it, obviously $\wp$ has double poles at each $\omega \in \Lambda$ and no other poles. Also, just as with the Θ-function, all elliptic functions can be expressed in terms of products of (powers of) translates of the σ-function, and one can write a formula for $\wp$ in this way: in fact,

$$\wp(z) - \wp(a) = -\frac{\sigma(z+a)\,\sigma(z-a)}{\sigma^2(z)\,\sigma^2(a)}. \quad \text{[Exercise]}$$

We now prove a different sort of "generation" result:

$\boxed{\text{Theorem 2}}$ Let $f \in \mathcal{M}er(\mathbb{C}/\Lambda)$. just another way of representing "ell. function w.r.t. Λ"

Then f may be expressed as a rational function in $\wp$ and $\wp'$. †

---

† Algebraist's version: $\mathcal{M}er(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp') \subseteq \dfrac{\mathbb{C}[x,y]}{\{\text{relation}\}}$ where x, y are indeterminates

Proof: f elliptic $\Longrightarrow$

$$f(z) = \underbrace{\frac{f(z)+f(-z)}{2}}_{\substack{\text{even}\\ \text{elliptic}}\ =: f_e} + \underbrace{\frac{f(z)-f(-z)}{2}}_{\substack{\text{odd}\\ \text{elliptic}}\ =: f_o} = f_e(z) + \underbrace{\frac{f_o(z)}{\wp'(z)}}_{\text{even elliptic}} \wp'(z)$$

So it suffices to treat even elliptic functions only, which we will show to be rational functions in $\wp$ alone. Assume f even.

Lemma: (a) $\mathrm{ord}_{z_0}(f) = m \Longrightarrow \mathrm{ord}_{-z_0}(f) = m$

(b) if $z_0 \underset{\Lambda}{\equiv} -z_0$, then $2 \mid \mathrm{ord}_{z_0}(f)$.

Proof // (a) for $m \geq 0$, $f^{(k)}(-z_0) = (-1)^k f^{(k)}(z_0)$.

for $m < 0$, look at $1/f$.

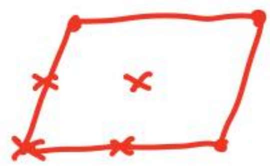
(b) z_0 is either 0, $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, or $\frac{\omega_1+\omega_2}{2}$ (the 4 2-torsion points of $\mathbb{C}/\Lambda$)

Assume $m \geq 0$ (otherwise use $1/f$).

if k is odd, $f^{(k)}$ is odd, and so

$-f^{(k)}(z_0) = f^{(k)}(-z_0) = f^{(k)}(z_0)$

$\Rightarrow f^{(k)}(z_0) = 0 \Rightarrow$ leading term in power-series expansion is $(z-z_0)^{\text{even}}$. //



Continuing the proof, let u_i $(i=1,\dots,r)$ be a family of points containing one representative from each class $\{u,-u\}$ mod Λ where f has a zero or pole, other than the class of Λ itself. Let

- $m_i := \mathrm{ord}_{u_i}(f)$, if $2u_i \underset{\Lambda}{\not\equiv} 0$, and
- $m_i := \frac{1}{2}\mathrm{ord}_{u_i}(f)$, if $2u_i \underset{\Lambda}{\equiv} 0$;

then $g(z) := \prod_{i=1}^{r} \left(\wp(z) - \wp(z_i)\right)^{m_i}$ has everywhere but $0 \pmod{\Lambda}$ the same order zeroes & poles as f. (The lemma applied to $\wp$ gives: for $z_0 \not\equiv_\Lambda -z_0$, $\wp(z) - \wp(z_0)$ has simple zeroes at z_0 & $-z_0$, & nowhere else $\pmod{\Lambda}$; for $z_0 \equiv_\Lambda -z_0$, $\wp(z) - \wp(z_0)$ has double zero at z_0, & nowhere else. One says that $0, \omega_1/2, \omega_2/2, \frac{\omega_1+\omega_2}{2}$ are the 4 <u>branch points</u> of $\wp$.) Hence g/f has no zeroes or poles away from 0, and so (by Lect. 22) it can't have 0/pole there either $\implies g/f$ constant. $\square$

III. The associated elliptic curve

Since $\mathbb{C}/\Lambda$ is 1-dimensional, one does not expect the transcendence degree of its field of meromorphic functions to exceed one. So the next step is to look for an <u>algebraic relation</u> between $\wp$ & $\wp'$. Write (in a neighborhood of 0)

- $$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \left\{ \frac{1}{\omega^2} \underbrace{\left(1 - \frac{z}{\omega}\right)^{-2}}_{\sum_{m \geq 0} (-1)^m \binom{-2}{m} \left\{\frac{z}{\omega}\right\}^m} - \frac{1}{\omega^2} \right\}$$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \sum_{m \geq 1} \frac{m+1}{\omega^2} \left\{\frac{z}{\omega}\right\}^m \qquad \frac{(-2)(-3)\cdots(-m-1)}{m!} = (-1)^m (m+1)$$

$$= \frac{1}{z^2} + \sum_{m \geq 1} \left\{(m+1)\, s_{m+2}(\Lambda)\right\} z^m = \frac{1}{z^2} + 3 s_4 z^2 + 5 s_6 z^4 + \cdots,$$

- $$\wp'(z) = -\frac{2}{z^3} + 6 s_4 z + 20 s_6 z^3 + \cdots$$

Now $$(\wp')^2 = \frac{4}{z^6} - \frac{24 s_4}{z^2} + \underbrace{\text{holo.}}_{-80 s_6 + \cdots}$$

$$4\wp^3 = \frac{4}{z^6} + \frac{36 s_4}{z^2} + \underbrace{\text{holo.}}_{60 s_6 + \cdots}$$

$$(\wp')^2 - 4\wp^3 = \frac{-60 s_4}{z^2} + \underbrace{\text{holo.}}_{-140 s_6 + \cdots}$$

So $(\wp')^2 - 4\wp^3 + 60 s_4 \wp$ = holo. elliptic $= C$.

From the constant terms alone, we clearly have

$$C = -140 s_6 .$$

We have proved the

$\boxed{\text{Theorem 3}}$ $(\wp')^2 = 4\wp^3 - g_2 \wp - g_3$,

where $g_2 = 60 s_4$ & $g_3 = 140 s_6$ depend on Λ.

$\boxed{\text{Corollary}}$ The map $\begin{cases} z \notin \Lambda \longmapsto (\wp(z), \wp'(z)) \\ \Lambda \longmapsto \text{"}\infty\text{"} \end{cases}$

parametrizing points on the nonsingular Weierstrass elliptic curve

complex variables

$$E_\Lambda := \left\{ (x, y) \;\middle|\; y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda) \right\} \cup \{\infty\}$$

actually gives an analytic isomorphism

$$\wp : \mathbb{C}/\Lambda \xrightarrow{\cong} E_\Lambda .$$

(think of as the Riemann surface of $\sqrt{4x^3 - g_2 x - g_3}$.)

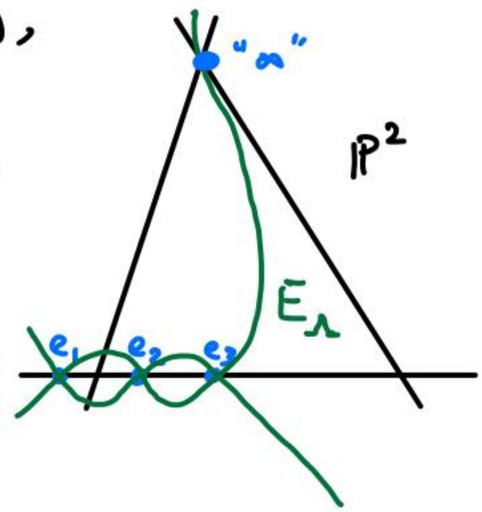
(complex 1-forms)

Proof: We will show P is 1-1 and onto:

(onto) E_Λ is connected & compact, as is $\mathbb{C}/\Lambda$ ($\Rightarrow$ map has closed image), so we are done by the open mapping theorem ($\Rightarrow$ map has open image).

(1-1) $\wp$ is the composition of P with projection $\left.\begin{matrix} E_\Lambda \to \mathbb{P}^1 \\ (x,y) \mapsto x \end{matrix}\right\}$.

Since the equation is quadratic in y, the projection has mapping degree 2. But mapping degrees multiply under composition, and the mapping degree of $\wp$ is 2. Since $\wp'$ is odd, the mapping degree of P is one. $\square$



More precisely,

$$E_\Lambda = \left\{ [X:Y:Z] \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; ZY^2 = 4X^3 - g_2XZ^2 - g_3Z^3 \right\}$$

and "∞" is $[0:1:0]$. Set

$$e_i := \wp\left(\frac{\omega_i}{2}\right),$$

where $\omega_3 := \omega_1 + \omega_2$. We know that $\wp(z) - \wp\left(\frac{\omega_i}{2}\right)$ has a zero of order 2 at $\frac{\omega_i}{2} = z$; so $\wp'(z)$ has a zero of order <u>one</u> there. But then

$$\frac{(\wp'(z))^2}{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}$$ is zero & pole-free

$\Longrightarrow$ constant,

and we conclude that

$$4x^3 - g_2 x - g_3 = 4\prod_{i=1}^{3}(x - e_i).$$

$\Rightarrow$ justify picture above

Since $(\wp')^2$ has only double zeroes, no two e_i can coincide, and so the discriminant of $\prod(x-e_i) = x^3 - \frac{g_2}{4}x - \frac{g_3}{4}$ must be nonzero:

$$0 \neq \prod_{i<j}(e_i - e_j)^2 = \begin{vmatrix} 1 & 0 & -\frac{g_2}{4} & -\frac{g_3}{4} & 0 \\ 0 & 1 & 0 & -\frac{g_2}{4} & -\frac{g_3}{4} \\ 3 & 0 & -\frac{g_2}{4} & 0 & 0 \\ 0 & 3 & 0 & -\frac{g_2}{4} & 0 \\ 0 & 0 & 3 & 0 & -\frac{g_2}{4} \end{vmatrix} = \frac{g_2^3 - 27g_3^2}{-4^3}$$

(a standard method of computing discriminant)

$$\Rightarrow \quad g_2^3 - 27g_3^2 \neq 0.$$

This will turn out to be an important quantity later.