Lecture 25: Introduction to modular forms

I. What is... a modular form?

Every complex 1-torus $\mathbb{C}/\mathbb{Z}\langle\omega_1,\omega_2\rangle$ is conformally isomorphic to one of the form $\mathbb{C}/\overbrace{\mathbb{Z}\langle 1,\tau\rangle}^{\text{"}\Lambda_\tau\text{"}}$ $\left(\tau = \frac{\omega_2}{\omega_1}\right)$, and henceforth we shall restrict our forms to this case. So far we have been taking τ to be fixed and varying $u \pmod{\Lambda}$; if we now let τ vary over the upper half-plane, we get a family of 1-tori $\{E_\tau\}_{\tau\in\mathfrak{h}}$. The modular group $\Gamma := SL_2(\mathbb{Z})$ (which is generated by $S := \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ & $T := \begin{pmatrix}1&1\\0&1\end{pmatrix}$) acts on this family by

$$\underset{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma}{\underbrace{\gamma}} . (\tau,u) = \Bigg(\underbrace{\frac{a\tau+b}{c\tau+d}}_{=:\gamma(\tau)}, \frac{u}{c\tau+d}\Bigg) ;$$

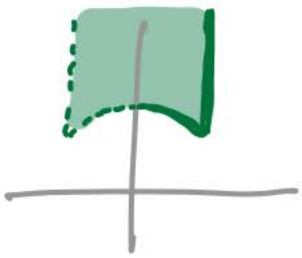
this is well-defined since for $u = m\tau+n \in \Lambda_\tau$ we get

$$\frac{u}{c\tau+d} = \frac{m\tau+n}{c\tau+d} \underset{\gamma\in\Gamma \Rightarrow \mathbb{Z}\langle a\tau+b, c\tau+d\rangle = \mathbb{Z}\langle 1,\tau\rangle}{=} m'\frac{a\tau+b}{c\tau+d} + n'\frac{c\tau+d}{c\tau+d} = m'\gamma(\tau)+n' \in \Lambda_{\gamma(\tau)} .$$

In fact, it follows at once from Theorem 1 of Lecture 21 that

$$\mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda_{\tau'} \iff \tau = \gamma(\tau') \text{ for some } \gamma \in SL_2(\mathbb{Z}),$$

so that

$\vec{\mathcal{F}}_{\Gamma}$ =  parametrizes isomorphism-classes of 1-tori in one-to-one fashion.

Restricting focus to the action on $\mathfrak{h}$, let $f \in \mathrm{Mer}(\mathfrak{h})$ and consider the automorphy property

$$(\Gamma)_k \qquad \boxed{f(\tau) = \frac{f(\gamma(\tau))}{(c\tau+d)^k} \left(=: f\big|^k_\gamma(\tau)\right) \quad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,}$$

which is equivalent to [Exercise: $(f|^k_\gamma)|^k_\eta = f|^k_{\gamma\eta}$ $\forall \gamma, \eta \in \Gamma$.]

$$\begin{cases} f(\tau+1) = f(\tau) \\ f(-1/\tau) = \tau^k f(\tau) . \end{cases}$$

The first of these implies

call this q

$$f(\tau) = F(e^{2\pi i \tau}) \quad \text{for some } F \in \mathrm{Mer}(D_1^*)$$

$$= \sum_{n=-\infty}^{\infty} a_n q^n ,$$

on a possibly smaller punctured disk (if f really has poles in $\mathfrak{h}$)

which is called the Fourier expansion of f.

Definition f is called a $\begin{cases} \text{meromorphic modular form} \\ \text{modular form} \\ \text{cusp form} \end{cases}$ of weight k (w.r.t. Γ) if

$$\begin{cases} a_n = 0 & \text{for } n < \text{some } N \in \mathbb{Z} \\ a_n = 0 & \text{for } n < 0 \text{ and } f \in \mathrm{Hol}(\mathfrak{h}) \\ a_n = 0 & \text{for } n \leq 0 \text{ and } f \in \mathrm{Hol}(\mathfrak{h}) \end{cases}$$

We write $\begin{cases} f \in \mathrm{Mer}_k(\Gamma) \\ f \in M_k(\Gamma) \\ f \in S_k(\Gamma) . \end{cases}$ ($\Longleftarrow$ clearly these are $\mathbb{C}$-vector spaces)

Remark// In view of "$(\Gamma)_k$" with $\gamma = -\mathrm{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, if k is odd then $f(\tau) = -f(\tau) \Rightarrow f$ is identically 0. So $\mathrm{Mer}_k(\Gamma)$, $M_k(\Gamma)$, $S_k(\Gamma)$ are $\{0\}$ for odd weight k.//

II. Examples of modular forms

Example 1// For $\Lambda = \mathbb{Z}\langle 1, \tau \rangle$, relabel $s_k(\Lambda)$ by $G_k(\tau)$ (which is more standard). That is,

$$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^k} .$$

We have

$$G_k(\tau+1) = \sum_{m,n}{}' \frac{1}{(m\tau+m+n)^k} \overset{\text{reindex}}{=} G_k(\tau) ,$$

$$G_k(-1/\tau) = \sum_{m,n}' \frac{1}{(-m/\tau + n)^k} = \tau^k \sum_{m,n}' \frac{1}{(-m+n\tau)^k} = \tau^k G_k(\tau),$$
reindex

and

$$\lim_{\tau \to i\infty} G_k(\tau) = \sum_{n \in \mathbb{Z}}' \frac{1}{n^k} = 2\zeta(k),$$ which $\Rightarrow a_0 = 2\zeta(k)$ & $a_n = 0$ ($\forall n < 0$)

$\Rightarrow$ $G_k \in M_k(\Gamma)$. The normalization

$$E_k(\tau) := \frac{1}{2\zeta(k)} G_k(\tau)$$

is called the Eisenstein series of weight k. //

Theorem 1 For $k \geq 4$ even,

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{(k-1)!\,\zeta(k)} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

(by previous evaluation of $\zeta(2m)$, this is $-2k/B_k$ ← Bernoulli)

where

$\sigma_{k-1}(n) := \sum_{\substack{m>0 \\ m \mid n}} m^{k-1}$ is the $(k-1)^{\text{st}}$ divisor function.

Proof: In Ahlfors's section on partial fractions he proves Euler's identity (for $\tau \in \mathbb{C} \setminus \mathbb{Z}$)

$$\sum_{n\in\mathbb{Z}}^{\text{P.V.}} \frac{1}{\tau+n} = \frac{1}{\tau} + \sum_{n\geq 1} \frac{2\tau}{\tau^2-n^2} = \pi \cot(\pi\tau)$$

Ahlfors p. 189, formula (11) presumably correct last term?

$$= \pi i \, \frac{e^{\pi i \tau}+e^{-\pi i\tau}}{e^{\pi i\tau}-e^{-\pi i\tau}} = -\pi i \, \frac{1+q}{1-q}$$

$$= -2\pi i \left(\frac{1}{2} + \sum_{r\geq 1} q^r \right).$$

$\tau \in \mathfrak{h}$

Differentiating $k-1$ times and dividing by $(-1)^{k-1}(k-1)!$ gives (using $\frac{d}{d\tau} q^r = \frac{d}{d\tau} e^{2\pi i r\tau} = 2\pi i r q^r$)

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r\geq 1} r^{k-1} q^r ,$$

and (k even)

$$G_k(\tau) = \sum_{n\in\mathbb{Z}}{}' \frac{1}{n^k} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{1}{(m\tau+n)^k}$$

$$= 2\zeta(k) + 2 \sum_{m\geq 1} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \underbrace{\sum_{m\geq 1} \sum_{r\geq 1} r^{k-1} q^{mr}}_{\sum_{n\geq 1} \sigma_{k-1}(n) q^n}$$

$\square$

Example 2 Theorem 1 yields

$$E_4(\tau) = 1 + 240q + 2160q^2 + \dots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - \dots$$

Moreover, the modular discriminant (which arose as the discriminant of the cubic E_τ)

$$\Delta(\tau) := \frac{g_2^3 - 27g_3^2}{(2\pi)^{12}} = \frac{1}{\underset{12^3}{1728}}\big(\underbrace{E_4^3 - E_6^2}_{\text{clearly has } a_0=0}\big)$$

$$= q - 24q^2 + \dots$$

Since E_4^3, E_6^2 both satisfy $(\Gamma)_{12}$, they belong to $M_{12}(\Gamma)$ and $\Delta \in S_{12}(\Gamma)$!!

Example 3 The Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{l\geq 1}(1-q^l) \in \mathrm{Hol}(\mathfrak{h})$$

clearly satisfies $\lim_{\tau\to i\infty}\eta(\tau) = 0$ and $\eta(\tau+1)^{24} = \eta(\tau)^{24}$.

In the next lecture we'll check that $\eta(-1/\tau)^{24} = \tau^{12}\eta(\tau)^{24}$.

$\Longrightarrow \eta^{24} \in S_{12}(\Gamma)$!

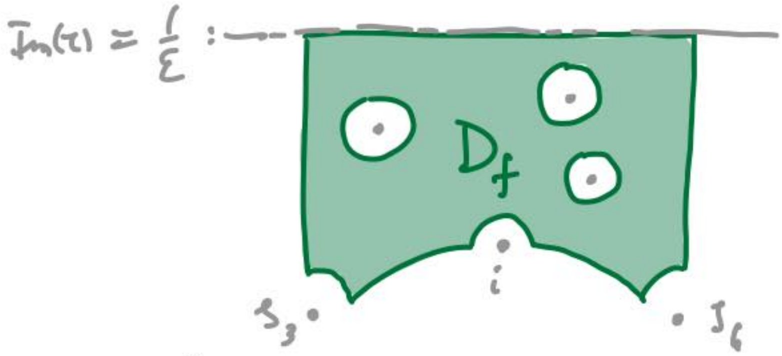
Two questions arise: do we have $\Delta \overset{(?)}{=} \eta^{24}$; and more generally how can we tell two modular forms are equal? It turns out that you only ever have to compute finitely many terms in the Fourier expansions to tell. This is because $M_k(\Gamma)$ is finite dimensional!

$\boxed{\text{Proposition}}$ Let $f \in M_k(\Gamma) \setminus \{0\}$; then writing $n_p = \begin{cases} 3 & \text{if } p = \zeta_6 \\ 2 & \text{if } p = i \\ 1 & \text{otherwise} \end{cases}$,

$$(\#) \quad \boxed{\sum_{p \in \mathcal{F}_\Gamma} \frac{1}{n_p} \operatorname{ord}_p(f) + \operatorname{ord}_\infty(f) = \frac{k}{12}.}$$

(1st n for which $a_n \neq 0$)

Proof: Let $D_f = \overline{\mathcal{F}_\Gamma} \setminus \left\{ \begin{array}{l} \varepsilon\text{-neighborhoods of all zeros of } f \\ \text{and the "nbhd. of } \infty\text{" } \operatorname{Im}(\tau) > \frac{1}{\varepsilon} \end{array} \right\}$:



Clearly $\int_{\partial D_f} d\log f = 0$. In the integral, the pieces over $\mathrm{Re} = \pm\frac{1}{2}$ cancel since $f(\tau) = f(\tau+1)$, and one gets fractional residues at $p = i$ and $p = \zeta_6$ (hence also at the "equivalent" point ζ_3). Since $d\log(f) = \mathrm{ord}_\infty(f)\,\frac{dq}{q} + (\text{holo. at } q=0)$, the $-\int_{-1/2+i/\varepsilon}^{1/2+i/\varepsilon}$ contributes $-2\pi i\,\mathrm{ord}_\infty(f)$. The tricky bit is the integral (in $\lim_{\varepsilon\to 0}$) over the bottom arc, excluding the tiny circles: using $f(-1/\tau) = \tau^k f(\tau) \implies d\log(f\circ S) = d\log f + k\frac{d\tau}{\tau}$,

$$\int_{\substack{\text{bottom}\\ \text{arc}}} d\log(f) = \int_i^{\zeta_6} d\log(f) - \int_i^{\zeta_6} d\log(f\circ S) = k\int_{\zeta_6}^{i}\frac{d\tau}{\tau}$$

$$= k\cdot\frac{\pi i}{6}.$$ All told, we get

$$0 = \left(-\sum \frac{1}{n_p}\,\mathrm{ord}_p(f) - \mathrm{ord}_\infty(f) + \frac{k}{12}\right)\cdot 2\pi i. \qquad \square$$

$\boxed{\text{Corollary 1}}$ $\dim M_k(\Gamma) = 0$ for k odd or negative, while for even $k \geq 0$,

$$\dim M_k(\Gamma) \leq \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv_{(12)} 2 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv_{(12)} 2 \end{cases}$$

and (for $k \geq 4$)

$$\dim S_k(\Gamma) = \dim M_k(\Gamma) - 1.$$

Proof: Set $m := \lfloor \frac{k}{12} \rfloor + 1$. Choose any distinct $p_1, \ldots, p_m \in \mathcal{F} \setminus \{i, \zeta_6\}$, and any $f_1, \ldots, f_{m+1} \in M_k(\Gamma)$. By linear algebra, $\exists\, f := \sum \alpha_i f_i$ (α_i not all 0) s.t. $f(p_i) = 0$ ($\forall i$). But then LHS (#) in the Proposition is $> \frac{k}{12}$, so $f \equiv 0$. Hence $\dim M_k(\Gamma) \leq m$.

Now, if $k \underset{(12)}{\equiv} 2$ then the only way to have $\sum \frac{1}{n_p} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = \frac{7}{6} + \text{integer}$ is to have (at least) a simple zero at i & a double zero at ζ_6 (which contribute $\frac{1}{2} + 2 \cdot \frac{1}{3} = \frac{7}{6}$). In addition, there are (at most) $\frac{k}{12} - \frac{7}{6} = \lfloor \frac{k}{12} \rfloor - 1 = m-2$ other zeroes (by the Prop.). Choosing $p_1, \ldots, p_{m-1} \in \mathcal{F} \setminus \{i, \zeta_6\}$ and $f_1, \ldots, f_m \in M_k(\Gamma)$, the above argument gives a relation on the f's $\Rightarrow \dim M_k(\Gamma) \leq m-1$.

The statement about $S_k(\Gamma)$ is clear from the existence of Eisenstein series (which by Theorem 1 are non-cuspidal). $\square$

$\boxed{\text{Corollary 2}}$ (i) $\dim M_{12}(\Gamma) = 2$ and $\dim S_{12}(\Gamma) = 1$

(ii) $f, g \in M_{12}(\Gamma)$ linearly independent $\Longrightarrow$

$f/g : \Gamma\backslash\mathfrak{h} \cup \{\infty\} \longrightarrow \mathbb{P}^1$ is an isomorphism.

Proof: Cor. 1 $\Rightarrow$ $\dim M_{12}(\Gamma) \le 2$, $\dim S_{12}(\Gamma) \le 1$.

The obvious independence of G_{12} & Δ (say) gives (i).

By the Prop., $\sum \frac{1}{n_p} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = 1$ $\Longrightarrow$ any

$af - bg$ has exactly one zero in $\overline{\mathcal{F}} \cup \{\infty\}$. So

(not both 0) $f/g \in \mathcal{M}er_0(\Gamma)^{\dagger}$ takes on every value

$b/a \in \mathbb{P}^1$ exactly once. $\square$

Example 4 (i) $\Rightarrow$ $\Delta = \eta^{24}$.

(ii) $\Rightarrow$ $j := \dfrac{E_4^3}{1728\,\Delta} = \dfrac{E_4^3}{E_4^3 - E_6^2} = \dfrac{g_2^3}{g_2^3 - 27 g_3^2}$

gives an isomorphism $j : \Gamma\backslash\mathfrak{h} \cup \{i\infty\} \longrightarrow \mathbb{P}^1$ sending

$i\infty \mapsto \infty$ hence $\Gamma\backslash\mathfrak{h} \xrightarrow{\sim} \mathbb{C}$. So the j-invariant

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$\dagger$ a weight-0 modular form is called a modular function. Obviously, quotients of modular forms of the same weight give meromorphic modular functions.

of an elliptic curve $y^2 = 4x^3 - g_2 x - g_3$ is a <u>precise</u> invariant of its isomorphism class. To learn a bit more about j, consider $E := \{ y^2 = 4x^3 \underset{\neq 0}{-g_3} \}$

automorphism of order 6 : $(x, y) \mapsto (\zeta_3 x, -y)$

and $E' := \{ y^2 = 4x^3 \underset{\neq 0}{-g_2 x} \}$.

automorphism of order 4 : $(x, y) \mapsto (-x, iy)$

Due to the existence of these <u>auto</u>morphisms, we must have

$$E \cong \mathbb{C}/\mathbb{Z}\langle 1, \zeta_6 \rangle \quad \text{and} \quad E' \cong \mathbb{C}/\mathbb{Z}\langle 1, i \rangle .$$

Conclude that

$$j(\zeta_6) = \frac{0}{0 - 27 g_3^2} = 0$$

and

$$j(i) = \frac{g_2^3}{g_2^3 - 0} = 1 .$$

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