Lecture 26: More on modular forms

I. Dedekind's eta function

Recall that we defined

$$\eta(\tau) := q^{1/24} \prod_{\ell \geq 1} (1-q^\ell) \in \mathrm{Hol}(\mathfrak{h}),$$

where $q := e^{2\pi i \tau}$ has $0 < |q| < 1$ for $\tau \in \mathfrak{h}$. Taking log,

$$\frac{1}{12}\pi i \tau - \log \eta(\tau) = -\sum_{\ell \geq 1} \log(1-q^\ell) = \sum_{k,\ell \geq 1} \frac{1}{k} q^{k\ell}$$

$$(*) \qquad = \sum_{k \geq 1} \frac{1}{k} \frac{q^k}{1-q^k}.$$

Set $h_n(z) := \cot\left((n+\tfrac{1}{2})\pi z\right) \cot\left(\frac{(n+\frac{1}{2})\pi z}{\tau}\right) / z$. This has:

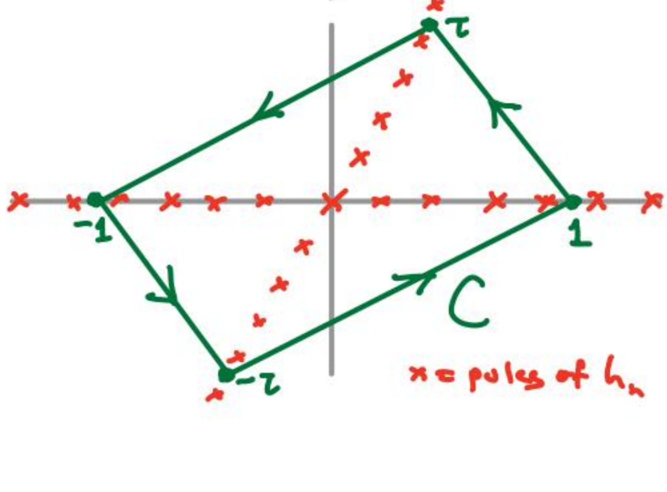
- Simple poles at $z = \dfrac{\pm k}{n+\frac{1}{2}}, \dfrac{\pm k\tau}{n+\frac{1}{2}}$

  with residues $\dfrac{1}{\pi k} \cot\left(\dfrac{\pi k}{\tau}\right)$ $(k = 1, 2, \ldots)$ resp. $\dfrac{1}{\pi k} \cot(\pi k \tau)$

- 3^{rd} order pole at $z = 0$ with residue $-\frac{1}{3}(\tau + \tau^{-1})$

  (use power series for sin & cos — totally routine).

Using the contour C



and the relation

$$-\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = 1 + \frac{2}{e^{-2iz}-1},$$

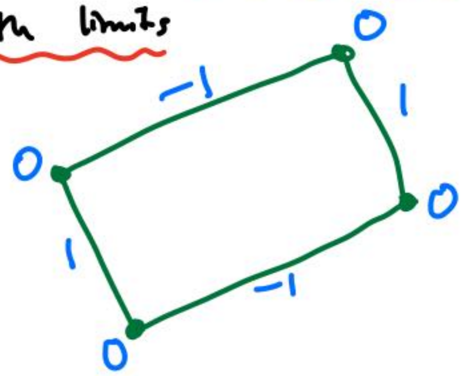
we therefore obtain:

$$\pi i \frac{\tau+\tau^{-1}}{12} + \frac{1}{8}\int_C h_n(z)\,dz = \frac{1}{2} i \sum_{k=1}^{n} \left(\frac{\cot(\pi k\tau)}{k} + \frac{\cot\left(\frac{\pi k}{\tau}\right)}{k} \right)$$

(**)

$$= \sum_{k=1}^{n} \frac{1}{k}\left(\frac{1}{e^{-2\pi i k\tau}-1} - \frac{1}{e^{2\pi i k/\tau}-1} \right)$$

Using the fact that $z h_n(z)$ is uniformly bounded on C as $n \to \infty$, with limits



[Exercise]

we have

$$\lim_{n\to\infty} \int_C h_n(z)\,dz = \left(\int_1^{\tau} - \int_{\tau}^{-1} + \int_1^{-\tau} - \int_{-\tau}^{1} \right) \frac{dz}{z}$$

$$= 4\log\tau - 2\log(-1) = 4\log(\tau/i).$$

Putting this together with (*) and (**), and setting $\tilde{q} = e^{-2\pi i/\tau}$,

$$\pi i \frac{\tau+\tau^{-1}}{12} + \frac{1}{2}\log\frac{\tau}{i} \underset{(**)}{=} \sum_{k\geq 1} \frac{1}{k}\frac{q^k}{1-q^k} - \sum_{k\geq 1}\frac{1}{k}\frac{\tilde{q}^k}{1-\tilde{q}^k}$$

$$\underset{(*)}{=} \frac{1}{12}\pi i\tau - \log\eta(\tau) - \frac{1}{12}\pi i(-\tau^{-1}) + \log\eta(-\tau^{-1})$$

$$\overset{\text{mult. by } 24}{\Longrightarrow} 24\log\eta(-\tau^{-1}) = 24\log\eta(\tau) + \log\tau^{12}$$

$$\Longrightarrow \boxed{\eta\left(-\frac{1}{\tau}\right)^{24} = \tau^{12}\,\eta(\tau)^{24}.}$$ So η^{24} is a cusp form of weight 12 w.r.t. Γ (see Example 3 of Lecture 25).

II. Ring structure

At the end of Lecture 16, we showed that

(#) $$j(\rho_6) = 0 \ , \quad j(i) = 1 \, .$$

Another way to see this is by using the Proposition (Lect. 25, p. 7).
For E_4, we clearly must have a simple zero at ρ_6;
while for E_6, we must have one at i. (#) follows.

Now here is a really amazing fact about E_4 & E_6.

$\boxed{\text{Theorem 1}}$ $\bigoplus_k M_k(\Gamma) = \mathbb{C}[E_4, E_6]$. That is, E_4 and E_6 freely generate the "graded ring of modular forms".

Proof: Let $P(X,Y) \in \mathbb{C}[X,Y]$ be a polynomial with

$$P(f_1(\tau), f_2(\tau)) \equiv 0$$

where f_1, f_2 are modular forms of the same weight.
By considering the weights, we see that

(##) $$P_d(f_1, f_2) \equiv 0$$

for each homogeneous component P_d of P. But

$$\frac{P_d(f_1, f_2)}{f_2^d} = p(f_1/f_2)$$

for some $p(t) \in \mathbb{C}[t]$. Since p has only finitely many roots, we can only have (**) if f_1/f_2 is a constant.

Consider the special case $f_1 = E_4^3$, $f_2 = E_6^2$. If we had $E_6^2 \equiv \lambda E_4^3$ for some $\lambda \in \mathbb{C}^*$, then $f := E_6/E_4$ would satisfy $\lim_{\tau \to i\infty} f(\tau) = 1$ and $f^2 = \frac{E_6^2}{E_4^2} = \lambda E_4$, hence $0 \not\equiv f \in M_2(\Gamma)$. But [Cor. 1, Lect. 25] says that $\dim M_2(\Gamma) = 0$. ※ So E_4^3 and E_6^2 can have no polynomial relation, and are therefore algebraically independent.

Furthermore, any algebraic relation $Q(E_4, E_6) \equiv 0$ breaks up into "homogeneous" components — summands with all terms of the same weight: $Q_k(E_4, E_6) \equiv 0$. Any two terms in such a summand, say $c E_4^a E_6^b$ and $c' E_4^{a'} E_6^{b'}$, have $4a + 6b = k = 4a' + 6b'$. But then

$$2a + 3b = 2a' + 3b' \implies a \underset{(3)}{\equiv} a',\ b \underset{(2)}{\equiv} b'$$

$$\implies \text{algebraic relation on } E_4^3, E_6^2. \text{ ※}$$

Finally, [Cor. 1, Lect. 25] says

(!) $$\boxed{\underset{\text{(even)}}{\dim M_k(\Gamma)} \le \begin{cases} \lfloor \frac{k}{12} \rfloor + 1, & k \underset{(12)}{\not\equiv} 2 \\ \lfloor \frac{k}{12} \rfloor, & k \underset{(12)}{\equiv} 2. \end{cases}}$$

For $k=12n$, the weight k part of $\mathbb{C}[E_4,E_6]$ is

$$\mathbb{C}\left\langle E_4^{3n}, E_4^{3(n-1)}E_6^2, E_4^{3(n-2)}E_6^4, \dots, E_6^{2n}\right\rangle$$

which has dimension $n+1$. For $k\in\{0,2,4,6,8,10\}$ it is $\mathbb{C}$, $\{0\}$, $\mathbb{C}\langle E_4\rangle$, $\mathbb{C}\langle E_6\rangle$, $\mathbb{C}\langle E_4^2\rangle$, and $\mathbb{C}\langle E_4\cdot E_6\rangle$. Putting these two calculations together gives $\dim(\underbrace{\mathbb{C}[E_4,E_6]_k}_{\text{weight } k \text{ part}})\geq \text{RHS}(!)$.
So in

$$\text{RHS}(!)\geq \dim M_k(\Gamma)\geq \dim \mathbb{C}[E_4,E_6]_k\geq \text{RHS}(!)$$

all inequalities are equalities and we are done. $\square$

Example 1// Comparing constant terms in Fourier expansions, we have (as a consequence of the Theorem)

$$E_4^2=E_8\,,\quad E_4E_6=E_{10}\,,$$
$$E_6E_8=E_4E_{10}=E_{14}.$$

This leads to some surprising number-theoretic identities. For instance,

$$E_4^2=1+480\sum_{n\geq1}\sigma_3(n)\,q^n+480\cdot120\sum_{n\geq1}\left(\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m)\right)q^n$$

and

$$E_8=1+480\sum_{n\geq1}\sigma_7(n)\,q^n$$

so

$$\boxed{\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m)=\frac{\sigma_7(n)-\sigma_3(n)}{120}\quad(\forall n)}.\quad //$$