Lecture 27: Modular forms of higher level

I. Level N

We now extend the definition of modular forms with respect to $\Gamma \left(= SL_2(\mathbb{Z})\right)$ to the "congruence subgroups"

$$\Gamma(N) := \ker\left\{ SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}) \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{(N)}{\equiv} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The intuitive idea is that $\Gamma(N)$ fixes the N^2 N-torsion sections of the family $\{E_\tau\}_{\tau \in \mathfrak{h}}$ (whereas Γ merely permutes these sections amongst themselves). Rather than proving this formally, we'll see it in the context of an example.

$\boxed{\text{Definition}}$ $f \in \mathcal{Hol}(\mathfrak{h})$ belongs to $M_k(\Gamma(N))$, i.e. is a modular form of weight k and level N, iff

- $f \equiv f|^k_\gamma \quad \forall\, \gamma \in \Gamma(N)$
- $\lim\limits_{\tau \to i\infty} (f|^k_\gamma)(\tau) < \infty \quad \forall\, \gamma \in \Gamma$.

What does the second bullet mean? First, for $\gamma = \text{id.}$, it's clearly true (since $\begin{pmatrix}1&N\\0&1\end{pmatrix} \in \Gamma(N) \Rightarrow f(\tau+N)=f(\tau)$)

then

$$f(\tau) = \sum_{\ell\in\mathbb{Z}} a_\ell q^{\ell/N},$$

and so this condition means that the a_ℓ are zero for negative ℓ. (Note that $q^{1/N}$ is a "local coordinate at ∞" on $\Gamma(N)\backslash\mathfrak{h}$.) Next, to get a fundamental domain for $\Gamma(N)$ you let $\gamma_1, \ldots, \gamma_M$ be a set of (coset) representatives of $\Gamma(N)\backslash\Gamma$, and write

$$\mathcal{F}_N := \bigcup_{i=1}^{M} \gamma_i(\mathcal{F}_\Gamma).$$

This will intersect $\mathbb{R}$ in finitely many points called cusps:

$$\kappa_N := (\overline{\mathcal{F}_N} \cap \mathbb{R}) \cup \{\infty\}.$$

or "$i\infty$"

For each $r \in \kappa_N \setminus \{\infty\}$, there is some (non-unique) $\gamma_r \in \Gamma$ with $\gamma_r(\infty) = r$ (e.g., amongst the $\{\gamma_i\}$). The second bullet in the definition, then, says that for each cusp r, the Fourier expansion (in $q^{1/N}$) of $f|^k_{\gamma_r}$ has no negative terms. Of course, as for Γ we can modify the above definition for cusp forms $\left(\lim_{\tau\to i\infty}(f|_\gamma^k)(\tau)=0\right)$ and meromorphic modular forms (left to reader).

Example 1 // If $\varphi:\mathbb{Z}^2\to\mathbb{C}$ is a function which

- is biperiodic with period N in each entry, and
- has $\varphi(0,0)=0$,

then the Eisenstein series

$$E_{k,\varphi}(\tau):=(\text{const.}*)\sum_{(m,n)\in\mathbb{Z}^2}\frac{\varphi(m,n)}{(m\tau+n)^k}$$

belongs to $M_k(\Gamma(N))$. [Exercise] //

Example 2 // Consider $f(\tau):=\wp_\tau\left(\frac{A\tau+B}{N}\right)$, where $\wp_\tau$ is the Weierstraß $\wp$-function for the lattice $\mathbb{Z}\langle 1,\tau\rangle$, and $A \& B\in\mathbb{Z}$ are not both $\equiv_{(N)} 0$. Let $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$.

Write

$$A\gamma(\tau)+B = A\frac{a\tau+b}{c\tau+d}+B = \frac{A'\tau+B'}{c\tau+d}$$

where $A'=Aa+Bc$, $B'=Ab+Bd$. Then

$$f(\gamma(\tau)) = \wp_{\gamma(\tau)}\left(\frac{A\gamma(\tau)+B}{N}\right)$$

$$= \frac{1}{\left(\frac{A'\tau+B'}{N(c\tau+d)}\right)^2} + \sum'_{(m,n)\in\mathbb{Z}^2}\left(\frac{1}{\left(\frac{A'\tau+B'}{N(c\tau+d)} - m\frac{a\tau+b}{c\tau+d} - n\right)^2} - \frac{1}{\left(m\frac{a\tau+b}{c\tau+d}+n\right)^2}\right)$$

$$= (c\tau+d)^2\left\{\frac{1}{\left(\frac{A'\tau+B'}{N}\right)^2} + \sum'_{(m',n')\in\mathbb{Z}^2}\left(\frac{1}{\left(\frac{A'\tau+B'}{N} - m'\tau - n'\right)^2} - \frac{1}{(m'\tau+n')^2}\right)\right\}$$

(unimodular transformation)

(!!) $$= (c\tau+d)^2\,\wp_\tau\left(\frac{A'\tau+B'}{N}\right)$$

$$= (c\tau+d)^2\,\wp_\tau\left(\frac{A\tau+B}{N}\right) = (c\tau+d)^2 f(\tau)$$

(IF $\frac{A'\tau+B'}{N} \equiv \frac{A\tau+B}{N}$ modulo $\Lambda_\tau := \mathbb{Z}\langle 1,\tau\rangle$. When does this happen? When $\begin{cases} A' = Aa+Bc \equiv A \mod N \\ B' = Ab+Bd \equiv B \mod N\end{cases}$, which happens for arbitrary A,B precisely when $a \underset{(N)}{\equiv} 1 \underset{(N)}{\equiv} d$, $b \underset{(N)}{\equiv} 0 \underset{(N)}{\equiv} c$, i.e. when $\gamma \in \Gamma(N)$!)

So we have checked the automorphy property for $f = f_{A,B}$. To check the growth conditions, note that by (!!) each $f_{A,B}\big|^2_\gamma = f_{A',B'}$ — so we really only need to look at the behavior as $\tau \to i\infty$. We have (for $A,B \in \{0,1,\dots,N-1\}$)

$$f_{A,B}(\tau) = \frac{1}{\left(\frac{A\tau+B}{N}\right)^2} + \sum_{m,n}{}' \left(\frac{1}{\Big(\underbrace{\tfrac{A\tau+B}{N} - m\tau - n}_{\text{rewrite as } \left(\frac{A}{N}-m\right)\tau + \left(\frac{B}{N}-n\right)}\Big)^2} - \frac{1}{(m\tau+n)^2} \right).$$

If $A \neq 0$ then all the $\frac{1}{\left(\frac{A\tau+B}{N} - m\tau - n\right)^2}$ terms $\longrightarrow 0$, so

$$\lim_{\tau \to i\infty} f_{A,B}(\tau) = -\sum_n{}' \frac{1}{n^2} = -2\zeta(2) = \frac{-\pi^2}{3}.$$

If $A = 0$ then the limit is $\left\{ \sum_n{}' \left(\frac{1}{\left(\frac{B}{N} - n\right)^2} - \frac{1}{n^2} \right) \right\} + \frac{1}{(B/N)^2}$.

Conclude that

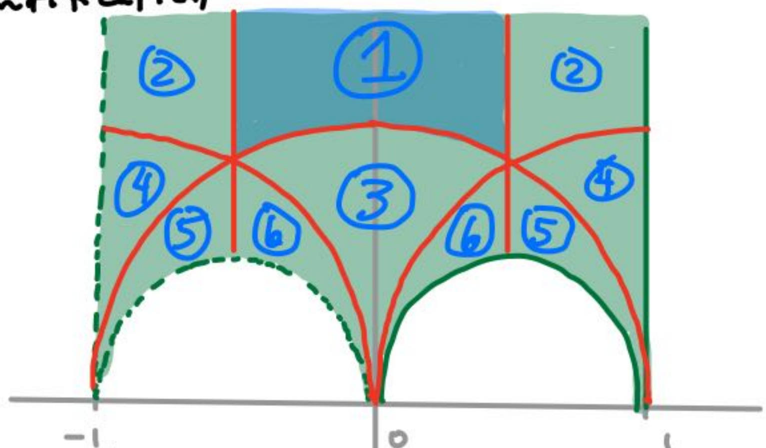
$$\boxed{f(\tau) \in M_2(\Gamma(N))}$$

So we can get modular forms of level N (and weight 2) by evaluating Weierstraß $\wp$ at N-torsion points. Similarly, we can get such forms by evaluating theta functions on N-torsion — these are called thetanulls.

II. Level two

The nicest-looking fundamental domain for $\Gamma(2)$ is actually obtained by chopping $\tilde{\mathcal{F}}_{\Gamma}$ into 2 pieces and orbiting them by 2 different sets of representatives of ${}_{\Gamma(2)}\backslash^{\Gamma}$ (cf. Ahlfors pp. 281-2, and note $|\Gamma(2)\backslash\Gamma| = 6$), which yields an identification

$$\Gamma(2)\backslash\mathfrak{h} \cong$$



where I have labeled (translated) "copies" of $\tilde{\mathcal{F}}_{\Gamma}$. We have $\kappa_2 = \{0, 1, \infty\}$, whereas $\Gamma\backslash\mathfrak{h}$ only had the cusp $\{\infty\}$.

Now, going back to Example 2, consider $f =$

$$\left.\begin{aligned} e_1(\tau) &:= \wp_\tau\left(\tfrac{1}{2}\right) \\ e_2(\tau) &:= \wp_\tau\left(\tfrac{\tau}{2}\right) \\ e_3(\tau) &:= \wp_\tau\left(\tfrac{\tau+1}{2}\right) \end{aligned}\right\} \text{ in (!!)}, \quad \frac{A'\tau+B'}{2} \underset{(\Lambda_\tau)}{\equiv} \left\{\begin{aligned} &1/2\,,\ \tau/2 \\ &(\tau+1)/2\,,\ 1/2 \\ &\tau/2\,,\ (\tau+1)/2 \end{aligned}\right.$$

(see Lemma 23)

for $\gamma = \underset{T}{\begin{pmatrix}1&1\\0&1\end{pmatrix}}, \underset{S}{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}$

So $(\ast\ast)$ $\begin{cases} e_1(\tau+1) = e_1\big|_T^2(\tau) = e_1(\tau) \\ e_2(\tau+1) = \cdots\cdots = e_3(\tau) \\ e_3(\tau+1) = \cdots\cdots = e_2(\tau) \end{cases}$ while $\begin{cases} e_1(\frac{-1}{\tau})/\tau^2 = e_1\big|_S^2(\tau) = e_2(\tau) \\ e_2(\frac{-1}{\tau})/\tau^2 = \cdots\cdots = e_1(\tau) \\ e_3(\frac{-1}{\tau})/\tau^2 = \cdots\cdots = e_3(\tau). \end{cases}$

By Example 2, we have $e_i \in M_2(\Gamma(2))$ $(i=1,2,3)$.

Note that the values of the cusp $\{\infty\}$ are given by

$$\lim_{\tau\to i\infty} e_2(\tau) = -\frac{\pi^2}{3} = \lim_{\tau\to i\infty} e_3(\tau) \quad \text{and}$$

$$\begin{aligned}\lim_{\tau\to i\infty} e_1(\tau) &= \frac{1}{(1/2)^2} + \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{1}{(n/2)^2} + \sum_{\substack{n\geq 3\\ n \text{ odd}}} \frac{1}{(n/2)^2} - 2\zeta(2) \\ &= 8\sum_{\substack{n \text{ odd}\\ n\geq 1}} \frac{1}{n^2} - 2\zeta(2) \\ &= 8\left(\frac{\pi^2}{8}\right) - \frac{\pi^2}{3} = \frac{2\pi^2}{3}\end{aligned} \Rightarrow \begin{cases} \lim_{\tau\to i\infty}(e_3-e_2) = 0 \\ \lim_{\tau\to i\infty}(e_1-e_2) = \pi^2. \end{cases}$$

Now consider

$$\lambda(\tau) := \frac{e_3-e_2}{e_1-e_2} \in \mathrm{Mer}_0(\Gamma(2)).$$

We have

$$\lambda(\tau+1) \overset{(\ast\ast)}{=} \frac{e_2-e_3}{e_1-e_3}(\tau) = \frac{e_3-e_2}{e_3-e_1} = \frac{e_3-e_2}{e_3-e_2-(e_1-e_2)} = \frac{\lambda(\tau)}{\lambda(\tau)-1}$$

$$\lambda(-1/\tau) \overset{(\ast\ast)}{=} \frac{e_3-e_1}{e_2-e_1}(\tau) = \frac{e_1-e_3}{e_1-e_2} = \frac{e_1-e_2-(e_3-e_2)}{e_1-e_2} = 1-\lambda(\tau).$$

So $\lim_{\tau \to i\infty} \lambda(\tau) = \frac{0}{\pi^2} = 0 \implies$

$$\lim_{\tau \to 0} \lambda(\tau) = \lim_{\tau \to i\infty} \lambda\left(-\frac{1}{\tau}\right) = \lim_{\tau \to i\infty} \left(1 - \lambda(\tau)\right) = 1 \ ,$$

$$\lim_{\tau \to 1} \lambda(\tau) = \lim_{\tau \to 0} \lambda(\tau+1) = \lim_{\tau \to 0} \frac{\lambda(\tau)}{\lambda(\tau)-1} = \infty .$$

Now recall that

$$\frac{1}{4}\left(\wp_\tau'(z)\right)^2 = \wp_\tau(z)^3 - \frac{g_2(\tau)}{4}\wp_\tau(z) - \frac{g_3(\tau)}{4}$$

$$= \Big(\wp_\tau(z) - \underbrace{\wp_\tau(1/2)}_{e_1}\Big)\Big(\wp_\tau(z) - \underbrace{\wp_\tau\left(\tfrac{\tau}{2}\right)}_{e_2}\Big)\Big(\wp_\tau(z) - \underbrace{\wp_\tau\left(\tfrac{\tau+1}{2}\right)}_{e_3}\Big)$$

(see Lecture 23)

$$\implies g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$$

$$g_3 = 4e_1e_2e_3$$

$$g_2^3 - 27g_3^2 = 4^2(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2$$

(cf. discriminant calculation in Lect. 23)

[which are compatible because $e_1+e_2+e_3=0$]

$$\implies j = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{-4(e_1e_2+e_2e_3+e_1e_3)^3}{(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2}$$

$$= \frac{4}{27}\frac{(1-\lambda-\lambda^2)^3}{\lambda^2(1-\lambda)^2} =: \phi(\lambda) .$$

— note: only sends $0, 1, \infty$ to ∞.

short calculation using $e_1+e_2+e_3=0$ and $\lambda = \frac{e_3-e_2}{e_1-e_2}$

[Exercise]

But now we have a commutative diagram

$$\begin{tikzcd}
\Gamma(2)\backslash\mathfrak{h} \arrow[r, "\lambda"] \arrow[d, two heads, "\pi"', "{}^{\dagger}\text{strict } 6:1"'] & \mathbb{P}^1\setminus\{0,1,\infty\} \arrow[d, two heads, "\phi", "6:1\ \text{(evident)}"'] \\
\Gamma\backslash\mathfrak{h} \arrow[r, "j", "\cong\ \text{(already knew this)}"'] & \mathbb{C}
\end{tikzcd}$$

If λ is not 1-1, it is generically m to 1 for $m>1$. But then $\phi\circ\lambda$ is generically $6m$-to-1, contradicting 6-to-1-ness of $j\circ\pi$ $(=\phi\circ\lambda)$.

Were λ not onto, $j\circ\pi$ could not be strictly 6-to-1.

So we arrive at

$\boxed{\textbf{Theorem}}$ $\underline{\lambda \text{ is an isomorphism.}}$

For higher level, the existence of an isomorphism from

$$\Gamma(N)\backslash\mathfrak{h} \cup \kappa_N \xrightarrow[\mu]{\cong} \mathbb{P}^1$$

(such a μ is called a Hauptmodul) is related to the number g_N at the end of Lecture 22 §II. There is one for $N=1,2,3,4,5$ only, because $N>5 \implies g_N>0$.

† this comes from the fact that $\Gamma/\Gamma(2) \cong \{id., S, T, ST, TS, STS\}$