Lecture 28: The Picard theorems

I. The monodromy theorem

A function element is a pair (f, U), U a region in $\mathbb{P}^1$ and $f \in \mathcal{H}ol(U)$. Two elements (f, U) & (g, V) are said to be direct analytic continuations of each other if $U \cap V \neq \emptyset$ and $f|_{U \cap V} \equiv g|_{U \cap V}$. This generates an equivalence relation,

$$(f, U) \equiv (g, V) \underset{\text{def.}}{\iff} \text{they are connected by a chain of direct analytic continuations.}$$

An equivalence class of function elements is called a global analytic function. Very roughly speaking, the point of view taken on Riemann surfaces in Chapter 8 of Ahlfors is that they are the "graphs" of these global analytic functions (obviously multivalued), "completed at branch points". Theorem 4 of this chapter gives a 1-to-1 identification between algebraic curves

$$\{(w, z) \mid P(w, z) = 0\} \subset \mathbb{C} \times \mathbb{C} \qquad (P \text{ polynomial})$$

and global analytic functions with finitely many sheets/branches and algebraic singularities.

The upshot is that "global analytic functions" give one notion of analytic continuation. In this section we review a more elementary one. Let $z_0 \in \mathbb{C}$, and take f to be a germ of an analytic function at z_0, i.e. a convergent power series centered there. Consider

$$\gamma : [a,b] \longrightarrow \mathbb{C} \qquad [\, C^0 \text{ path with } \gamma(a) = z_0 \,]$$

and choose

$$(\ast) \begin{cases} \Pi = \{ \underset{a}{\overset{\shortparallel}{a_0}}, a_1, \ldots, \underset{b}{\overset{\shortparallel}{a_{n+1}}} \} \text{ a partition of } [a,b] \quad a < \cdots < b \\ \{D_i\}_{i=0}^{n} \text{ disks with } D_i \supset \gamma([a_i, a_{i+1}]). \end{cases}$$

If there exist $f_i \in \mathrm{Hol}(D_i)$ with $f_i|_{D_i \cap D_{i+1}} \equiv f_{i+1}|_{D_i \cap D_{i+1}}$ $(\forall i)$ for any choice $(\ast)$, we say that f admits analytic continuation along γ and call the power series of f_n at $\gamma(b)$ the "terminal germ". (Denote this by g for now.)

Proposition The terminal germ is independent of the choices $(\ast)$.

Proof: We need only check that we can choose Π and the $\{D_i\}$ "independently" without affecting g. If we pick different D_i', then the "common refinement" $D_i'' := D_i \cap D_i'$

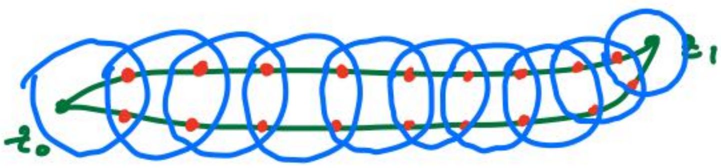
still produces a continuation, and it's clear that it is the restriction of the other two. (Crucial here is that the D_i" are, while not disks, connected.) If we add some $c \in (a_k, a_{k+1})$ to Π, then we may choose (f_k, D_k) for both subintervals $[a_k, c]$ and $[c, a_{k+1}]$. Done. $\square$

$\boxed{\text{Theorem 1 ("The monodromy theorem")}}$ Let $\Omega \subset \mathbb{P}^1$ be a connected open set; and take f to be an analytic germ at $z_0 \in \Omega$, admitting analytic continuation along any path in Ω. Then given any two paths $\gamma, \eta \subset \Omega$ from z_0 to $z_1 \in \Omega$, homotopic in Ω, the terminal germs f_γ and f_η are equal.

Proof: Let $\Psi : [a,b] \times [0,1] \to \Omega$ be a homotopy (with $\Psi(a,s) = z_0$, $\Psi(b,s) = z_1$ $(\forall s)$, and $\Psi(t,0) = \gamma$, $\Psi(t,1) = \eta$). Using the uniform continuity of Ψ (and the finite distance from image(Ψ) to $\partial\Omega$), there exists a partition

$$0 = s_0 < s_1 < \cdots < s_m = 1$$

such that successive $\Psi(t, s_j)$ are "close together": that is, there exists a choice of the a_i and $D_i \subset \Omega$ such that $\Psi([a_i, a_{i+1}], s_j)$ and $\Psi([a_i, a_{i+1}], s_{j+1})$ are contained in D_i.



The Theorem follows (using the Proposition, of course). $\square$

Example // The dilogarithm

$$\mathrm{Li}_2(z) := \sum_{k \geq 1} \frac{z^k}{k^2}$$

exhibits interesting "monodromy" (= continuation properties, often around a "singular point" through which continuation is impossible). Continued once around $z = 1$, it becomes

$$\mathrm{Li}_2(z) + 2\pi i \cdot \log(z),$$

which is no longer analytic at 0!! If γ denotes the commutator $\gamma_1^{-1}\gamma_0^{-1}\gamma_1\gamma_0$

then the continuation along γ goes

$$\overset{f}{\mathrm{Li}_2(z)} \xmapsto{\gamma_0} \mathrm{Li}_2(z) \xmapsto{\gamma_1} \mathrm{Li}_2(z) - 2\pi i \log(z)$$
$$\xmapsto{\gamma_0^{-1}} \mathrm{Li}_2(z) - 2\pi i \left(\log(z) - 2\pi i\right)$$
$$\xmapsto{\gamma_1^{-1}} \mathrm{Li}_2(z) + 2\pi i \log(z) - 2\pi i \left(\log(z) - 2\pi i\right) = \overset{\gamma}{\mathrm{Li}_2(z)} - 4\pi^2.$$

This path is homologous but not homotopic to zero in $\mathbb{C} \setminus \{0,1\} = \Omega$, which emphasizes the importance of "homotopic" in the Theorem. //

II. Little Picard

We now use the monodromy theorem and one product of our discussion of modular forms (the λ function) to prove the following major result.

Theorem 2 $f \in \mathrm{Hol}(\mathbb{C})$ with $f(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$ (distinct) is constant.

Proof: Replacing f by $\dfrac{f(z) - a}{b - a}$, it suffices to do the case $a = 1$, $b = 0$, so that $f(\mathbb{C}) \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$ ... which happens to be the range of the λ-function! There exists a $\tau_0 \in \mathfrak{h}$ (say, in $\mathcal{F}_2$) with $\lambda(\tau_0) = f(0)$; and since λ is 1-1 on $\bar{\mathcal{F}}_2$, $\lambda'(\tau_0) \neq 0$. So there is a local inverse $\lambda_0^{-1} : \Delta_0 \to \mathfrak{h}$, when $\Delta_0 \ni f(0)$ is a small disk. Take Ω_0 to be the component of $f^{-1}(\Delta_0)$ containing 0, and set

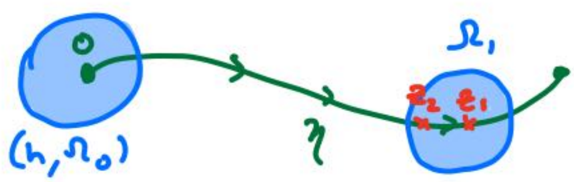
$$h := \lambda_0^{-1} \circ f \; : \; \Omega_0 \longrightarrow \lambda_0^{-1}(\Delta_0) \subset \mathfrak{h}.$$
$$0 \longmapsto \tau_0$$

In a diagram:

$$\begin{tikzcd}
 & \mathfrak{h} \arrow[d, "\lambda"] & \supset & \lambda_0^{-1}(\Delta_0) \ni \tau_0 \\
\mathbb{C} \arrow[ur, squiggly, "\text{want to construct}"] \arrow[r, "f"] & \mathbb{C}\setminus\{0,1\} & \supset & \Delta_0 \ni f(0) \arrow[u, dashed, "\lambda_0^{-1}"'] \\
\Omega_0 \arrow[u, phantom, "\subset"] \arrow[urrr, dashed, bend right, "(f)"] \arrow[uurrr, dashed, bend left, "h"] & & &
\end{tikzcd}$$

We need something stronger than "h can be analytically continued along all paths in $\mathbb{C}$" — we want this AND that all analytic continuations remain with image in $\mathfrak{h}$.

Suppose otherwise: then there exists a path η



where analytic continuation is not possible past z_1 OR $\mathrm{Im}(h)$ becomes non-positive there. Now, $\exists\, \tau_1 \in \mathfrak{h}$ s.t. $\lambda(\tau_1) = f(z_1)$, and repeating the above construction gives Δ_1, λ_1^{-1}, Ω_1, and $\tilde{h}_1 := \lambda_1^{-1} \circ f : \Omega_1 \to \mathfrak{h}$. Denoting the analytic continuation of h (to z_2) by h, we have (as z_2 "varies to the left of z_1" in $\eta \cap \Omega_1$)

$$\lambda(\tilde{h}_1(z_2)) = f(z_2) = \lambda(h(z_2)).$$

Hence, for some $\gamma \in \Gamma(2)$, we have in a neighborhood of z_2

$$h_1 := \gamma \circ \tilde{h}_1 = h.$$

(A priori, we could have a different γ for each z_2, s.t. $\gamma(\tilde{h}_1(z_2)) = h(z_2)$; but $\tilde{h}_1$ and h vary continuously and γ cannot.) But then h_1 continues h past z_1 with $\mathrm{Im}(h_1) > 0$, a contradiction.

Therefore h continues along every path in $\mathbb{C}$, with positive imaginary part. By the monodromy theorem and simple-connectedness of $\mathbb{C}$, it continues to an entire function

$$H : \mathbb{C} \longrightarrow \mathfrak{h}.$$

But then e^{iH} is bounded, hence (by Liouville) constant.
So H hence $f = \lambda \circ H$ is constant. $\square$

III. Montel's normality criterion

This will be required for the proof of "big Picard".

Theorem 3 Let $\Omega \subset \mathbb{C}$ be a connected open set, $a \,\&\, b \in \mathbb{C}$ distinct, $\mathcal{F} := \{f \in \mathcal{H}ol(\Omega) \mid f(\Omega) \subset \mathbb{C} \setminus \{a,b\}\}$. Then $\mathcal{F}$ is a normal family "in the classical sense": any sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence which either converges normally or tends "to the constant ∞" uniformly on compact subsets.

Remarks: (a) As above, we may assume $a=1$, $b=0$.

(b) We may assume $\Omega = D_1$. (Normality is a local property: any Ω is covered with a countable collection of balls, and if any $\{f_n\} \subset \mathcal{F}$ has a subsequence converging on D_1 normally in this sense, a subsubsequence converges normally on $D_1 \cup D_2$, etc., then the diagonal subsequence converges normally on all disks; and any compact K can be written as a union of finitely many compact $K_i \subset D_i$.)

(c) We may replace $\mathcal{F}$ by $\mathcal{F}_1 := \{f \in \mathcal{F} \mid |f(0)| \leq 1\}$. (Given $\{f_n\} \subset \mathcal{F}$, either a subsequence lies in $\mathcal{F}_1$ or in $1/\mathcal{F}_1$. Suppose we are in the latter setting: i.e. $1/f_{n_k} \in \mathcal{F}_1$ and we have proved the result for $\mathcal{F}_1$.

Since the f_{n_k} don't have 0 in their range, $1/f_{n_k}$ are holomorphic; and by Hurwitz, a subsequence of the f_{n_k} either go normally to 0, a nowhere-zero holomorphic function, or ∞. Therefore the same applies to $1/f_{n_k}$.)

(d) If $\mathcal{G}$ is a set of functions, holomorphic on D_1, <u>with range in $\mathfrak{h}$</u>, then $\mathcal{G}$ is normal in the classical sense. This was a HW problem, but I give the proof anyway:

Given $\{g_n\} \subset \mathcal{G}$, $|e^{ig_n}| < 1 \Rightarrow e^{ig_n}$ holo. + uniformly bounded $\overset{\text{usual}}{\underset{\text{Montel}}{\Longrightarrow}}$ $\exists$ normally convergent subsequence $\exp(ig_{n_k})$, with uniform limit 0 or e^{ig} (nowhere 0) by Hurwitz. (Here g exists b/c D is simply connected.) In the first case, $g_{n_k} \xrightarrow[\text{on compact}]{\text{unif.}}$ "$i\infty$"; in the second, there are actually 2 possibilities. Let K be compact; then $e^{ig(K)} \subset$ compact annulus $\Rightarrow g(K)$ is contained in a compact rectangle in $\mathfrak{h}$. One has from the uniform convergence $e^{ig_{n_\ell}|_K} \to e^{ig|_K}$ that $g_{n_\ell}|_K$ goes "uniformly to $g|_K$ mod $2\pi\mathbb{Z}$". From this, <u>either</u> $g_{n_\ell}|_K$ has a subsequence $\xrightarrow[\text{unif.}]{} g|_K + m$ <u>OR</u> a subsequence with $\pm \operatorname{Re}(g_{n_{\ell_k}}) \xrightarrow[\text{unif}]{} \infty$. $\square$

<u>Proof of Theorem 3:</u> So assume

$$\mathcal{F}_1 := \{ f \in \mathcal{H}ol(D_1) \mid f(D_1) \subset \mathbb{C} \setminus \{0,1\},\ |f(0)| \le 1 \}.$$

We need to produce a (classically) normally convergent subsequence of a given $\{f_n\} \subset \mathcal{F}_1$. Since $\{|f_n(0)|\} \subset \overline{D}_1$, $\exists \{f_{n_k}\}$ s.t. $f_{n_k}(0) \to \mu \in \overline{D}_1$.

$\boxed{\text{CASE 1 } (\mu \neq 0,1)}$ Fixing a branch of λ^{-1} in a neighborhood of μ, define $\hat{f}_{n_k} :=$ analytic continuation of $\lambda^{-1} \circ f_{n_k}$ (a priori defined in nbhd. of 0) as in §II. Since $\hat{f}_{n_k} : D_1 \to \mathfrak{h}$, by Remark (d) $\exists\, \hat{f}_{n_{k_j}}$ converging normally to a (necessarily analytic, nonzero †) limit function g. A priori, $g(D_1) \subset \overline{\mathfrak{h}}$; but since $g(0) = \lambda^{-1}(\mu) \in \mathfrak{h}$ and D_1 is open, the open mapping thm. $\Rightarrow$ $g(D_1) \subset \mathfrak{h} \Rightarrow \lambda \circ g$ defined, and $f_{n_{k_j}} = \lambda \circ \hat{f}_{n_{k_j}} \xrightarrow[j \to \infty]{\text{normally}} \lambda \circ g$.

$\boxed{\text{CASE 2 } (\mu = 1)}$ Let $h_k = \sqrt{f_{n_k}}$ be chosen so that $h_k(0) \to -1$ as $k \to \infty$. (We can take $\sqrt{\ }$ b/c D_1 is simply connected, and f_{n_k} is nowhere 0.) Then h_k omits 0 & 1, $|h_k(0)| \leq 1$, $h_k(D_1) \subset \mathbb{C} \setminus \{0,1\}$. So we're back in Case 1. Squaring the normally convergent $h_{k_j} \Rightarrow f_{k_j}$ conv. normally.

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† it can't be $\equiv 0$ or $\equiv \infty$ b/c $\mu \in \overline{D}_1^{\,*}$.

$\boxed{\text{CASE 3 } (\mu = 0)}$ Let $\tilde{h}_k = 1 - f_{n_k}$; then in Case 2.

This completes the proof. $\square$

One can see this Montel theorem as an example of the heuristic (& in general false) "Bloch principle" that properties which force entire functions to be constant also force families of functions to be normal.

IV. Big Picard

Theorem 4 Let $U \subset \mathbb{C}$ be a connected open set containing z_0, and $f \in \mathrm{Hol}(U \setminus \{z_0\})$ have an essential singularity at z_0. Then f omits at most one value $a \in \mathbb{C}$.

Corollary In a neighborhood of an essential singularity, an analytic function takes every complex value, with possibly one exception, infinitely often.

Proof of Corollary (assuming Theorem): Take $z_1 \in U$ where $f(z_1) = b \neq a$. (This exists by the Theorem.) Then take a disk about z_0 excluding z_1; applying Theorem 4 again, $\exists\, z_2$ in that disk where $f(z_2) = b$; and so on. $\square$

Note that the Corollary strengthens Casorati–Weierstraß, which merely asserts that the range of f was dense in $\mathbb{C}$. The Theorem also implies Little Picard, since in that result the case of an essential singularity at ∞ is the only nontrivial one. (Otherwise we can use the fundamental theorem of algebra, since $f \in \mathrm{Hol}(\mathbb{C})$ with a pole at ∞ is a polynomial.)

Proof of Theorem 4: wolog assume $U = D_1$, $z_0 = 0$, f omits $a, b \in \mathbb{C}$, & show 0 is a pole or removable singularity of f.

Suppose 0 is NOT a pole: i.e., $\exists\, M$ and $z_n \to 0$ s.t. $|f(z_n)| \le M$ $(\forall n)$. There exists a subsequence $f(z_{n_k}) \xrightarrow[k\to\infty]{} w \in \mathbb{C}$. We may assume $|z_{n_k}|$ strictly decreasing and that $|z_{n_k}| < \frac{1}{2}$ $(\forall k)$. Set $\lambda_k := 2z_{n_k}$, and consider the sequence of functions

$$f_k : D_1^* \longrightarrow \mathbb{C} \setminus \{a, b\}$$

defined by $f_k(z) := f(\lambda_k z)$. By Theorem 3, passing to a subsequence if necessary, f_k converges normally to a holomorphic function or to ∞. In fact, it can't converge to ∞, because $f_k(1/2) = f(z_{n_k}) \to w$. Write F for the limit function.

Now put $K_k := \max\limits_{|z| = \frac{1}{2}} |f_k(z)| \left(= \max\limits_{|z| = |z_{n_k}|} |f(z)| \right)$.

Since $f_k \xrightarrow[\text{unif.}]{} F$ on the compact set $|z| = \frac{1}{2}$, and $\max_{|z|=\frac{1}{2}} |f(z)| < \infty$, we get $K := \sup_{k \in \mathbb{N}} K_k < \infty$.

By the MMP, however, we have

$$\|f(z)\|_{\bar{A}(z_{n_k}, z_{n_{k-1}})} \leq \max\{K_{k-1}, K_k\} \leq K$$

annulus $|z_{n_k}| \leq |z| \leq |z_{n_{k-1}}|$

$$\Rightarrow \|f(z)\|_{D^*_{1/2}} \leq K$$

$\Rightarrow$ 0 is a removable singularity,

done. $\square$