Lecture 29: Geometric function theory

I. Overview

The next few lectures will be devoted to a topic known as geometric function theory, which has interesting connections with the topics studied so far this term: normal families, analytic continuation, Riemann Mapping Thm., etc.

To get a feeling for the flavor of this area, consider $f \in \mathrm{Hol}(D)$ $(D := D_1)$ with $f(0) = 0$ and $f'(0) = 1$.

We might ask:

"Schlicht functions"

(1) Is there a "minimal" subdisk of the domain or range where f must be injective (i.e. 1-1)?

(2) If f is injective on all of D, are there upper bounds on the power-series coefficients? (Note that $f(z) = z + \sum_{n \geq 2} a_n z^n$.)

(BONUS) In both cases, what are the "extremal functions" like?

The answer to the second question, as we shall see, is "yes" simply because the power series coefficients are continuous functions on a compact set (of functions) in the topology of normal convergence. That the actual best bounds are $|a_n| \le n$ is the Bieberbach Conjecture (1916) proved by L. de Branges (1985). That the family of schlicht functions is normal (or rather, its proof) will become part of the proof of the Normalization Theorem for Riemann surfaces, which we shall do at the end of this segment of the course.

The answer to the first question is also yes, which we will prove below, but the "best bounds" are only conjectured. I should point out that A. Baernstein, a professor in our department until his death in 2014, made groundbreaking contributions to both problems.

II. Bounded functions

Let's first restrict to a case of the first problem where the family is obviously normal hence solutions to extremal problems guaranteed.

Set $\mathcal{F} := \{ f \in \mathrm{Hol}(\mathbb{D}) \mid f(0) = 0,\ f'(0) = 1 \}$,

$\overline{\mathcal{F}} := \{ f \in \mathrm{Hol}(\overline{\mathbb{D}}) \mid f(0) = 0,\ f'(0) = 1 \}$.

<u>Lemma 1</u>: $f \in \overline{\mathcal{F}},\ \|f\|_{\mathbb{D}} \le M \implies M \ge 1,\ f(\mathbb{D}) \supseteq \mathbb{D}_{\frac{1}{6M}}$.

Proof: By the Cauchy inequalities,

$$|a_n| \le \lim_{r \to 1^-} \frac{\|f\|_{\partial \mathbb{D}_r}}{r^n} \le M \quad (\forall n)$$

$$\underset{a_1 = 1}{\implies} M \ge 1.$$

For $z \in \partial \mathbb{D}_{1/4M}$, we have

$$|f(z)| \ge |z| - \sum_{n \ge 2} |a_n| |z|^n$$

$$\ge \frac{1}{4M} - \sum_{n \ge 2} \frac{M}{(4M)^n} \rightarrow \frac{M/(4M)^2}{1 - 1/4M} = \frac{1}{4(4M-1)}$$

$$= \frac{1}{4} \left(\frac{(4M-1) - M}{M(4M-1)} \right) \rightarrow \frac{3M-1}{4M-1} \ge \frac{2}{3}$$

$$\ge 1/6M.$$

Set $g(z) := f(z) - w$ for any $w \in D_{1/6M}$. Then

$$|f(z) - g(z)| = |w| < \frac{1}{6M} \le |f(z)|$$

$\underset{\text{Rouché !}}{\Longrightarrow}$ f and g have the same # of zeros in $D_{1/4M}$

$\underset{f(0)=0}{\Longrightarrow}$ $g(z_0) = 0$ for some $z_0 \in D_{1/4M}$

$\Longrightarrow$ $w \in f(D_{1/4M}) \subset f(D)$. $\square$

Rescaling gives

$\boxed{\text{Proposition 1}}$ $g \in \mathcal{H}ol(D_R)$, $g(0) = 0$, $|g'(0)| = \mu > 0$, and $\|g\|_{D_R} \le M$ $\Longrightarrow$ $g(D_R) \supset D_{R^2\mu^2/6M}$.

Proof: We have $f(z) := \dfrac{g(Rz)}{Rg'(0)} \in \mathcal{S}_{se}$, with $\|f\|_D \le \dfrac{M}{\mu R}$.

By Lemma 1, $f(D) \supset D_{\mu R/6M}$; the conclusion follows. $\square$

We will also need (for Wednesday's lecture)

Lemma 2: $f \in \mathcal{H}ol(D(\alpha, r))$ s.t.

(*) $\boxed{|f'(z) - f'(\alpha)| < |f'(\alpha)| \quad (\forall z \in D^*(\alpha, r))}$

$\Longrightarrow$ f is injective.

Proof: Given $\gamma = \underbrace{[z_1 \overset{\neq}{,} z_2]}_{\text{segment}} \subset D(a, r)$,

$$\begin{aligned} |f(z_1) - f(z_2)| &\underset{\text{FTC}}{=} \left| \int_\gamma f'(z)\, dz \right| \\ &\underset{\Delta\text{ ineq.}}{\geq} \left| \int_\gamma f'(a)\, dz \right| - \left| \int_\gamma \{f'(z) - f'(a)\}\, dz \right| \\ &\geq |f'(a)||z_1 - z_2| - \int_\gamma |f'(z) - f'(a)|\, ds \\ &\underset{\text{by } (*)}{>} 0 \end{aligned}$$

$\Longrightarrow f(z_1) \neq f(z_2)$. $\square$

On Wednesday we will prove

Bloch's Theorem: $f \in \mathcal{F} \Longrightarrow \exists$ disk $S \subset D$ such that $f|_S$ is injective and $f(S)$ contains a disk of radius $\frac{1}{72}$.