Lecture 3 : Extension to the boundary

The potential ugliness of the boundary of a simply-connected region $\Omega \subsetneq \mathbb{C}$ was historically a major obstacle to the late 19^{th} Century attempts (via "Dirichlet Principle"/potential theory) to prove the full RMT.

What we'll discuss today are situations in which the map $f : \Omega \xrightarrow{\cong} D_1$ produced by RMT admits a continuous resp. analytic extension to the closure $\overline{\Omega}$ (or part of it).

At the end I'll give one more proof of the Riemann Mapping Theorem itself, one with a more constructive & dynamical flavor. For reference, recall the statement:

<u>RMT</u> : For $\Omega \subsetneq \mathbb{C}$ a simply connected region, there exists a biholomorphic mapping $f : \Omega \to D_1$.

I. A topological result

Let $\Omega, \Omega' \subseteq \mathbb{C}$ be regions.

$\boxed{\text{Definition}}$ A sequence $\{z_n\} \subset \Omega$ (resp. path $\gamma : [0,1] \to \Omega$) is said to approach the boundary of Ω "$\to \partial\Omega$" iff for each compact $K \subset \Omega$ there exists $N \in \mathbb{N}$ (resp. $\epsilon > 0$) such that $\{z_n\}_{n \geq N} \subset \Omega \setminus K$ (resp. $\gamma((1-\epsilon, 1)) \subset \Omega \setminus K$).

$\boxed{\text{Proposition}}$ Let $F : \Omega \to \Omega'$ be a homeomorphism, and $\{z_n\}$ (or $\gamma(t)$) $\longrightarrow \partial\Omega$. Then $\{F(z_n)\}$ (or $F(\gamma(t))$) $\to \partial\Omega'$.

Proof: Let $K' \subset \Omega'$ be compact. Since F is a homeomorphism, $F^{-1}(K')$ is compact. But then some tail of $\{z_n\}$ or $\gamma(t)$ stays outside $F^{-1}(K')$, and so the corresponding tail of $\{F(z_n)\}$ or $F(\gamma(t))$ avoids K'. $\square$

II. Extension via Schwarz reflection

Let Ω be as in the RMT.

Definition (a) An <u>analytic arc</u> is a map

$$\gamma : (a,b) \longrightarrow \mathbb{C}$$

which is 1-to-1 and real-analytic, with γ' nowhere 0.

[As usual we shall write γ for the image as well.]

(b) A (free, 1-sided) <u>analytic boundary arc of Ω</u> is an analytic arc $\gamma \subset \partial\Omega$ with (complex-) analytic extension

$$\tilde{\gamma} : \Delta \rightarrow \mathbb{C}$$

where :
- $\Delta \subset \mathbb{C}$ is open & simply-connected, with $\Delta \cap \mathbb{R} = (a,b)$
- Δ is symmetric under complex conjugation
- $\tilde{\gamma}^{-1}(\Omega) = \Delta \cap \mathfrak{h}$.

Note that the extension $\tilde{\gamma}$ is unique, and defined by the same power-series defining γ.

Now let $f : \Omega \rightarrow D_1$ be as in the RMT.

Theorem <u>If $\gamma \subset \partial\Omega$ is an analytic boundary arc, there is an extension $\tilde{f} \in \mathcal{H}ol(\Omega \cup \gamma)$ with $\tilde{f}(\gamma)$ an analytic boundary arc of ∂D_1.</u>

Proof: WOLOG assume
- $\tilde{\gamma}'$ nowhere 0
- $\tilde{\gamma}$ 1-to-1

by shrinking Δ, so that $\tilde{\gamma} : \Delta \to \tilde{\gamma}(\Delta)$ is a conformal isomorphism. It will suffice to extend

$$F := f \circ (\tilde{\gamma}|_{\Delta \cap \mathfrak{h}}) \in \mathrm{Hol}(\Delta \cap \mathfrak{h})$$

to $\tilde{F} \in \mathrm{Hol}(\Delta)$, since

$$\tilde{f} := \tilde{F} \circ \tilde{\gamma}^{-1} \in \mathrm{Hol}(\tilde{\gamma}(\Delta))$$

then agrees with f on $\tilde{\gamma}(\Delta \cap \mathfrak{h})$.

Shrinking Δ further if necessary, we may assume $0 \notin F(\Delta \cap \mathfrak{h})$, so that $i \log F$ is defined on $\Delta \cap \mathfrak{h}$. Since for any $\{z_j\} \subseteq \Delta \cap \mathfrak{h}$ approaching $\mathbb{R}$ we have $F(z_j) \to \partial D_1$ by the Proposition above (§I), $\mathrm{Im}\,(i \log F(z_j)) = \log |F(z_j)| \longrightarrow 0$. By Schwarz reflection, we may extend $i \log F(z)$

- by its limit on $(a,b) = \Delta \cap \mathbb{R}$
- by $\overline{i \log F(\bar{z})}$ on $\Delta \cap (-\mathfrak{h})$

to a holomorphic function on Δ. Taking $\exp(-i(\cdot))$ yields the extension of F itself (which satisfies $i \log \tilde{F}(\bar{z}) = -i \log \overline{\tilde{F}(z)} = i \log \left(\frac{1}{\overline{\tilde{F}(z)}}\right) \Rightarrow \tilde{F}(\bar{z}) = 1/\overline{\tilde{F}(z)}$).

It remains to check that $\tilde{f} \circ \gamma = \tilde{F}|_{(a,b)}$ is 1-to-1.

If $\tilde{F}'(x_0) = 0$ for any $x_0 \in (a,b)$, then $\tilde{F}$ would have to map not just $x_0 \pm \epsilon$ but curves with tangent $x_0 \pm \epsilon e^{i\pi/n}$ (for some $n \geq 2$) to ∂D_1, impossible since $\tilde{F}(\Delta \cap h) \cap \partial D_1 = \emptyset$ (and such curves <u>would</u> intersect $\Delta \cap h$). But if $\tilde{F}'(x_0) \neq 0$, then

$$0 > \frac{\partial \log|\tilde{F}|}{\partial y}\Big|_{x_0} \underset{\text{CR eqns.}}{=\!=} -\frac{\partial \arg \tilde{F}}{\partial x}\Big|_{x_0} \Longrightarrow$$

$\tilde{F}|_{(a,b)}$ moves strictly counterclockwise in ∂D_1 as x increases, hence is 1-to-1. $\square$

<u>Note</u>: If you forgot about Schwarz reflection, it's on pp. 172-3 of Ahlfors.

III. Caratheodory's Theorem

The next result concerns the case where the boundary of Ω is a continuous Jordan curve: i.e. there is a C^0 map

unit circle in $\mathbb{C}$

$$\gamma : S^1 \longrightarrow \partial\Omega$$

that is 1-1 & onto, hence a homeomorphism. (Ω is taken to be the bounded component of $\mathbb{C}\setminus\gamma(S^1)$, and is a bounded, simply connected region.)

Theorem Let $\varphi : D_1 \to \Omega$ be a conformal isomorphism, with Ω as above (bounded, simply-connected region, with $\partial\Omega$ C^0 Jordan). Then there exists $\hat{\varphi} : \overline{D}_1 \to \overline{\Omega}$ C^0 & 1-to-1, such that $\hat{\varphi}\big|_{D_1} = \varphi$ — that is, φ admits an extension to a homeomorphism of the (compact) closures.

The proof is long and is deferred to Lecture 4.
An obvious Corollary of this Theorem (together w/RMT) is that for Ω_1, Ω_2 Jordan-curve-bounded regions, $\exists$ homeomorphism $\overline{\Omega}_1 \xrightarrow{\cong} \overline{\Omega}_2$ restricting to a conformal isomorphism $\Omega_1 \xrightarrow{\cong} \Omega_2$.

IV. The second proof of RMT

We can construct approximate mappings of a bounded simply-connected region Ω into D_1 in the sense of the following

Lemma (Caratheodory): $\exists \{f_n\} \subset Hol(\Omega, D_1)$ s.t.

(a) $f_n(P) = 0$ (for some fixed $P \in \Omega$)

(b) $f_n(z)$ maps Ω to a region Ω_n in 1-1 fashion, with $D_{r_n} \subseteq \Omega_n \subseteq D_1$ ($r_n \in (0,1)$).

(c) $r_n \longrightarrow 1$ as $n \to \infty$.

(Somewhat heuristic) Proof: Take $\Omega_0 = \Omega$, and define $\Omega_1 := f_1(\Omega_0)$, where $f_1(z) := \kappa \cdot (z - P)$ translates & dilates Ω_0 to fit it inside D_1. Let

$r_1 :=$ radius of the largest $D_r \subset \Omega_1$.

Some $z_1 \in \partial D_{r_1}$ is not in Ω_1 (since Ω_1^c is closed and $d(\Omega_1^c, \partial D_{r_1}) = 0$).

Now inductively define, given $\begin{cases} \Omega \xrightarrow{f_n} D_1 \quad \text{w/ image} =: \Omega_n \\ z_n \in \partial\Omega_n \end{cases}$,

- f_{n+1} by $\left(\frac{z_n - f_n(z)}{1-\bar{z}_n f_n(z)}\right)^{1/2} = \frac{z_n^{1/2} - f_{n+1}(z)}{1-\overline{z_n^{1/2}} f_{n+1}(z)}$
- $\Omega_{n+1} := f_{n+1}(\Omega)$
- $r_{n+1} :=$ radius of largest D_r in Ω_{n+1}
- $z_{n+1} \notin \Omega_{n+1}$ with $|z_{n+1}| = r_{n+1}$ (may not be unique).

recall that $\tilde{\phi}_\alpha(z) := \frac{\alpha - z}{1-\bar{\alpha} z}$

Notice that $f_{n+1} = \left(\tilde{\phi}^{-1}_{\sqrt{z_n}} \circ S^{-1} \circ \tilde{\phi}_{z_n}\right) \circ f_n$:

preserves D_1

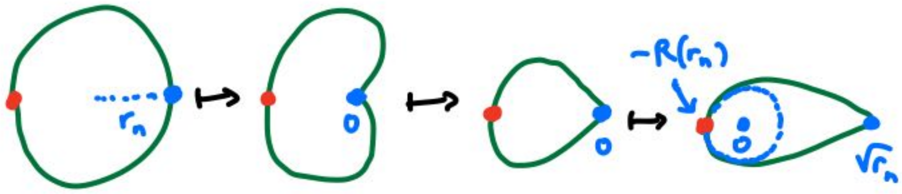
$\sqrt{}$ is well-def'd. on $\tilde{\phi}_{z_n}(\Omega_n)$

geometrically, $\tilde{\phi}^{-1}_{\sqrt{z_n}} \circ S^{-1} \circ \tilde{\phi}_{z_n}$ pushes the point z_n on the boundary of Ω_n to zero, takes square root, and pushes 0 out again, to a point at distance $r_n^{1/2}$ from 0 (i.e. further out). The circle of radius r_n is mapped to a lemniscate (w/interior) of smallest radius $=: R(r_n)$, and clearly this is a lower bound for r_{n+1}.

To compute r_{n+1}, it is enough to consider

$$\tilde{\phi}^{-1}_{\sqrt{r_n}} \circ S^{-1} \circ \tilde{\phi}_{r_n},$$

which sends



(It's up to you to check that the closest point indeed has phase π.)

We have $$\sqrt{\frac{r_n(1-e^{i\pi})}{1-r_n^2 e^{i\pi}}} = \frac{\sqrt{r_n} - R(r_n)e^{i\pi}}{1-\sqrt{r_n}\,R(r_n)e^{i\pi}}$$

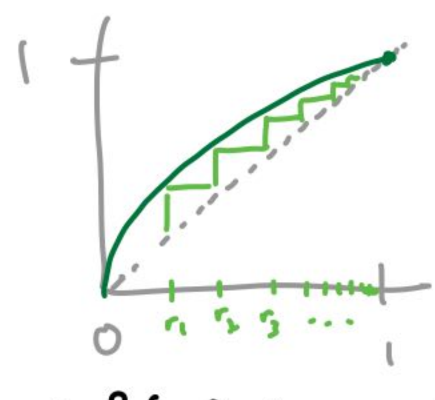
$$\Rightarrow \quad \sqrt{\frac{2r_n}{1+r_n^2}} = \frac{\sqrt{r_n} + R(r_n)}{1+\sqrt{r_n}\,R(r_n)}$$

$$\Rightarrow \quad R(r_n) = \frac{\sqrt{r_n}\,(r_n - 1) + \sqrt{2r_n(1+r_n^2)}}{1+r_n} \quad (\leq r_{n+1}).$$

(The point is that if the r_n-disk is contained in Ω_n, then the lemniscate I drew has to be in Ω_{n+1}. So...) Locally at 0, $R(r)$ has dominant term $(\sqrt{2}-1)\sqrt{r}$ $(> r)$; thus for small r the function $R(\cdot)$ is increasing fast. Further, solving

$$r = \frac{\sqrt{r}\,(r-1) + \sqrt{2r(1+r^2)}}{1+r}$$ leads to $r(r^2-1)(\sqrt{r}-1)^2 = 0$

meaning the graph of R looks like



so clearly $r_n < R(r_n) \leq r_{n+1} \leq 1$

and $r_n \longrightarrow 1$. $\square$

SECOND PROOF of RMT:

We'll do this for bounded simply-connected Ω — obviously enough (cf. the proof of Lemma 2 in Lecture 2):

- The $\{f_n\}$ produced by the Lemma are uniformly bounded by 1; and so by Montel, some subsequence converges uniformly on (all) compact subsets. Since the limit function f is (on any such set) a uniform limit of analytic functions, it must be analytic. Furthermore, $f(p)=0$.
- The same argument as in the 1^{st} proof shows f is 1-1.
- So (as before) we must check f is onto D_1. This goes a little differently:

Take any $w_0 \in D_1$; w_0 lies in $D_{1-2\epsilon}$ for some $\epsilon > 0$. We may assume Ω is bounded and obtain that the limit F of $\{f_n^{-1} \mid n \geq N\}$ (choose N s.t. $r_N \geq 1-\epsilon$) on $D_{1-\epsilon}$ is analytic and 1-1 (by taking a subsequence and applying Montel & Hurwitz as above). Consider the compact subset $F(\overline{D}(w_0,\epsilon)) \subset \Omega$: since F is 1-1, the interior $F(D(w_0,\epsilon))$ is a neighborhood of $F(w_0)$ (with compact closure $\subset \Omega$), and

so contains all $\{f_m^{-1}(w_0)\}$ for $m \geq M$ $(\geq N)$. Also
(some large integer)

the $\{f_n\}$ converge uniformly on the compact closure (and are continuous there), so that we may write

$$w_0 = \lim_{n\to\infty} (f_n \circ f_n^{-1})(w_0) = \lim_{n\to\infty} \lim_{m\to\infty} (f_n \circ f_m^{-1})(w_0)$$

$$= \lim_{n\to\infty} f_n\left(\lim_{m\to\infty} f_m^{-1}(w_0)\right) = \lim_{n\to\infty} f_n(F(w_0)) = f(F(w_0))$$

so that indeed f hits w_0. $\square$