Lecture 30: The Bloch and Landau constants

In Lecture 29 we proved two results which we will need to reference today:

Proposition 1 $g \in \mathcal{H}ol(D_R)$, $g(0) = 0$, $|g'(0)| = \mu > 0$, and $\|g\|_{D_R} \le M \implies g(D_R) \supset D_{R^2\mu^2/6M}$.

and

Lemma 2: $f \in \mathcal{H}ol(D(\alpha, r))$ s.t.

$$(*) \quad \boxed{|f'(z) - f'(\alpha)| < |f'(\alpha)| \quad (\forall z \in D^*(\alpha, r))}$$

$\implies f$ is injective.

We will also need the notation

$$\mathcal{F} := \{f \in \mathcal{H}ol(D) \mid f(0) = 0,\ f'(0) = 1\},$$

$$\overline{\mathcal{F}} := \{f \in \mathcal{H}ol(\overline{D}) \mid f(0) = 0,\ f'(0) = 1\}.$$

I. Bloch's† Theorem

Given $f \in \mathcal{F}$ with <u>no</u> bound, we would like to find some disk (not necessarily centered about zero) in the domain &/or range so that f is injective there.

Start by assuming $f \in \overline{\mathcal{F}}$. Put (for $r \in [0,1]$)

$K(r) := \|f'\|_{\partial D_r} = \|f'\|_{\overline{D_r}}$;

$h(r) = (1-r) \cdot K(r) \in C^0([0,1])$, with $h(0) = 1$, $h(1) = 0$;

$r_0 := \sup\{r \mid h(r) = 1\} \in [0,1)$.

Let $\alpha \in \partial D_{r_0}$ be such that $|f'(\alpha)| = K(r_0) \left(= \frac{h(r_0)}{1-r_0}\right)$; then

(**) $$\boxed{|f'(\alpha)| = \frac{1}{1-r_0}.}$$

Set $\rho_0 := \frac{1-r_0}{2}$, let $z \in D(\alpha, \rho_0) \left(\subset D_{\frac{1+r_0}{2}}\right)$; then

$$|f'(z)| \leq \|f'\|_{D_{\frac{1+r_0}{2}}} = K\left(\tfrac{1}{2}(1+r_0)\right) = \frac{h\left(\frac{1}{2}(1+r_0)\right)}{1-\frac{1}{2}(1+r_0)}$$

(***) $$< \frac{1}{1-\frac{1}{2}(1+r_0)} = \frac{1}{\rho_0}$$

(by defn. of r_0 & $\frac{1+r_0}{2} > r_0$)

† André Bloch, not to be confused with Spencer. Entire output produced in mental asylum after murdering his family.

$$\underset{\text{using }(**)}{\Longrightarrow} \quad |f'(z)-f'(\alpha)| \le |f'(z)| + |f'(\alpha)|$$

$$< \frac{1}{\rho_0} + \frac{1}{1-r_0} = \frac{3}{2\rho_0}$$

$$\Longrightarrow \quad |f'(z)-f'(\alpha)| < \frac{3|z-\alpha|}{2\rho_0^2}$$

using Schwarz: writing $z = \zeta\rho_0 + \alpha$, $\zeta = \frac{z-\alpha}{\rho_0}$, we apply it to $F(\zeta) := \frac{2\rho_0}{3}\{f'(\zeta\rho_0+\alpha) - f'(\alpha)\} \in \mathcal{H}ol(D)$, which has $F(0)=0$, $|F| \le 1$ $\underset{\text{Schwarz}}{\Longrightarrow} |F(\zeta)| \le |\zeta|$

$$\Longrightarrow |f'(z)-f'(\alpha)| \le \frac{3}{2\rho_0}\cdot\frac{|z-\alpha|}{\rho_0}.$$

$\Longrightarrow$ for $z \in S := D(\alpha, \frac{\rho_0}{3})$,

$$|f'(z)-f'(\alpha)| < \frac{3\cdot\rho_0/3}{2\rho_0^2} = \frac{1}{2\rho_0} = |f'(\alpha)|$$

$\underset{\text{Lemma 2}}{\Longrightarrow}$ $f|_S$ is injective.

Now define $R := \frac{\rho_0}{3}$, $\mu := \frac{1}{2\rho_0}$, $M := \frac{1}{3}$,

$$g(z) := f(z+\alpha) - f(\alpha) : D_R \longrightarrow \mathbb{C}, \quad 0 \longmapsto 0,$$

and note $|g'(0)| = |f'(\alpha)| = \frac{1}{2\rho_0} = \mu$. For $z \in D_R$,

$$\gamma := [\alpha, z+\alpha] \subset S \subset D(\alpha, \rho_0)$$

$$\Longrightarrow |g(z)| = \left|\int_\gamma f'(w)\,dw\right| \underset{(***)}{\le} \frac{\overset{<\rho_0/3}{|z|}}{\rho_0} < \frac{1}{3} = M.$$

$$\underset{\text{Proposition 1}}{\Longrightarrow} g(D_R) \supset D_\sigma, \text{ where } \sigma := \frac{R^2\mu^2}{6M} = \frac{(\rho_0^2/9)(1/4\rho_0^2)}{6/3} = \frac{1}{72}$$

$\Longrightarrow \quad f(S) \supset D(f(\alpha), \frac{1}{72})$.

So for $f \in \overline{\mathcal{F}}$, we have what we want.

What if $f \in \mathcal{H}ol(\overline{D_r})$? Setting

$F(z) := \dfrac{f(rz) - f(0)}{rf'(0)}$, we have $F \in \overline{\mathcal{F}}$ hence (for some disk $S_0 \subset D$)

$F(S_0) \supset$ disk of radius $\frac{1}{72}$, $F|_{S_0}$ injective

$\Longrightarrow \exists$ disk $S_1 \subset D_r$ s.t.

(†) $\boxed{f(S_1) \supset \left\{\text{disk of radius } \frac{r|f'(0)|}{72}\right\}, \ f|_{S_1} \text{ injective}}$

So if $f \in \mathcal{F}$, we can look (for $s \in (0,1)$) at

$$f_s(z) := \frac{f(sz)}{s} \in \mathcal{H}ol(D_{\frac{1}{s}}) \subset \mathcal{H}ol(\overline{D_{\frac{1+s}{2s}}})$$

$\underset{\text{by (†)}}{\Longrightarrow} \exists$ disk $S_s \subset D_{\frac{1+s}{2s}}$ on which f_s is 1-1, with image containing a disk Δ_s of radius $\frac{s+1}{2s \cdot 72}$.

But then $f|_{\underset{\underset{D_1}{\cap}}{sS_s}}$ is 1-1, with image $s\Delta_s$ of radius $\frac{s+1}{2 \cdot 72}$.

Looking back at the argument, one sees that the choice of α in the construction of S_s can be chosen continuously so that $s_1 < s_2 < 1 \Longrightarrow$

$$s_1 S_{s_1} \subset s_2 S_{s_2} \ (\subset D_1), \quad s_1 \Delta_{s_1} \subset s_2 \Delta_{s_2} \ (\subset f(D_1)).$$

So taking $s \to 1^-$, we obtain a disk of radius $\frac{1+1}{2 \cdot 72} = \frac{1}{72}$ in the injective image of a subdisk of D, and hence the

Theorem (Bloch, 1925) $f \in \mathcal{F} \implies \exists$ disk $S \subset D$ such that $f|_S$ is injective and $f(S)$ contains a disk of radius $\frac{1}{72}$.

II. The two constants

The constant $\frac{1}{72}$ is actually a terrible lower bound; what's important is that there exists one at all, so that in the following B is not zero:

Definition 1 Given $f \in \mathcal{F}$, let

$$\beta(f) := \sup \left\{ r \,\middle|\, \begin{array}{l} \exists \text{ disk } S \subset D \text{ s.t. } f|_S \text{ is 1-1} \\ \text{and } f(S) \supset \text{disk of radius } r \end{array} \right\}.$$

Then Bloch's constant is

$$B := \inf \{ \beta(f) \mid f \in \mathcal{F} \} \quad \left(\geq \tfrac{1}{72}\right).$$

$\boxed{\text{Definition 2}}$ Given $f \in \mathcal{F}$, let

$$\lambda(f) := \sup\{r \mid f(D) \supset \text{disk of radius } r\}.$$

Then Landau's constant is

$$L := \inf\{\lambda(f) \mid f \in \mathcal{F}\} \ (\geq B).$$

One question that arises is: what is the meaning of this "sup"? Is there actually a disk of radius L in the image of $f \in \mathcal{F}$, or just a sequence of disks with radii approaching L?

$\boxed{\text{Proposition 2}}$ $f \in \mathcal{F} \Rightarrow f(D) \supset \left(\begin{matrix}\text{disk of radius} \\ \lambda(f)\end{matrix}\right)$ $(\geq L)$

Proof:
- sufficient to do for $f \in \overline{\mathcal{F}}$
- use compactness of $f(\overline{D})$

Rest is left as an exercise. $\square$

What is known?

$$0.43 < B < 0.47 \quad , \quad 0.5 < L < 0.55.$$

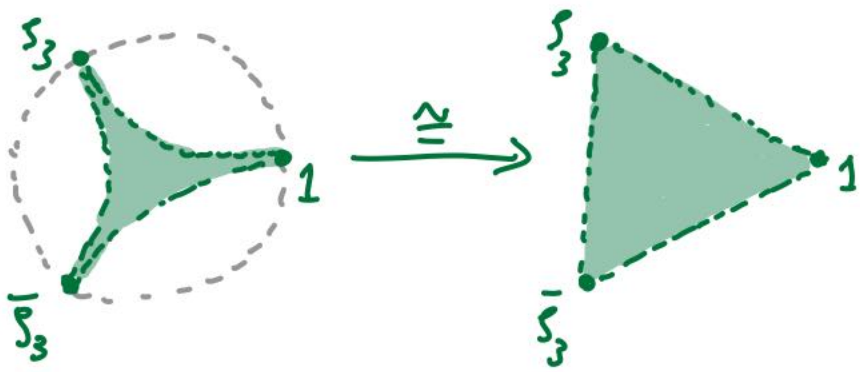
In fact,

$$B \leq B_0 := \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})(1+\sqrt{3})^{1/2}} \quad , \quad L \leq L_0 := \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{2})}.$$

$\boxed{\text{Conjecture}}$ $\underline{B = B_0, \quad L = L_0.}$

There is a known function in each case having these "proposed extremal" values of β resp. λ.

Example 1 // Begin with the conformal isomorphism of open regions (guaranteed by RMT)



and extend by Schwarz reflection repeatedly to obtain a function

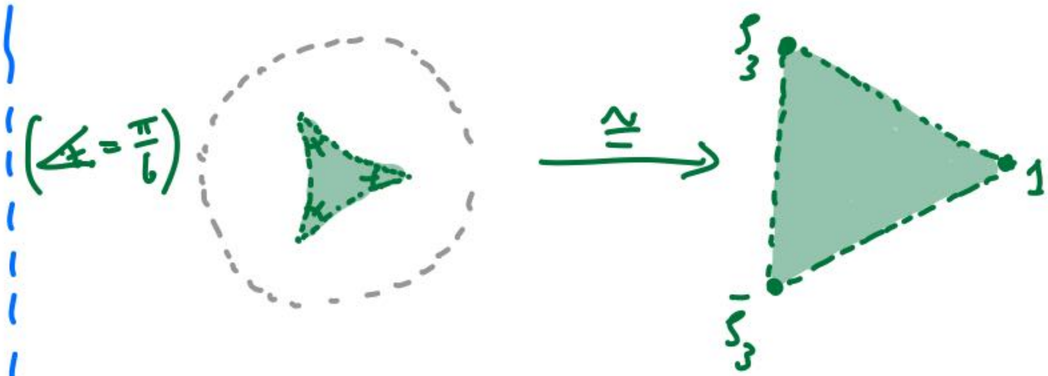
$$f_1 : D \twoheadrightarrow \mathbb{C} \setminus \mathcal{A},$$

where $\mathcal{A} \subset \mathbb{C}$ is the lattice given by $1 + \mathbb{Z}\langle \zeta_3 - 1, \bar{\zeta}_3 - 1 \rangle$.

In fact, f_1 is a covering map, i.e. a surjection which is everywhere a local isomorphism (no "branching" $z \mapsto z^k$), exactly like $\lambda : \mathfrak{h} \to \mathbb{C} \setminus \{0,1\}$. In both cases ($f_1$ & λ) the domain is simply connected and so this is

called the universal cover (of $\mathbb{C}\setminus\Lambda$ resp. $\mathbb{C}\setminus\{0,1\}$).
It turns out that $f_1'(0) = L_0^{-1}$, while D_1 is the largest disk in the image $\mathbb{C}\setminus\Lambda$. //

Example 2 // Begin with the RMT-guaranteed CE



and extend by Schwarz reflection to get

$$f_2 : \mathbb{D} \twoheadrightarrow \mathbb{C},$$

which is this time a branched cover. The largest disk in the image which is the 1-1 image of a disk is (again) D, and this time it so happens that

$f_2'(0) = B_0^{-1}$. //

There is work by Baernstein & Vinson checking minimality of $\lambda(f_1)$ $(= L_0)$ amongst similarly constructed functions (i.e. universal coverings of $\mathbb{C}\setminus\{$lattice which is "close to hexagonal"$\}$).

III. Smale's mean-value conjecture

The following problem arose in the context of applying Newton's method to determine roots of polynomials:

Given $P(z) = z + \sum_{j=2}^{n+1} a_j z^j$, let $z_1, \dots, z_n$ be the zeroes of $P'(z)$, and $w_j := P(z_j)$. Put

$$\sigma(P) := \min_{1 \le j \le n} \frac{|w_j|}{|z_j|} .$$

Conjecture (Smale) $\sigma(P) \le \frac{n}{n+1}$, $n \ge 1$.

Theorem (Czarzynski–Kotkowski 1978; Smale 1981) $\sigma(P) \le 4$

This will be a consequence of one of the results for schlicht/univalent functions we prove next (Köbe $\frac{1}{4}$ theorem).