# Lecture 31: Schlicht functions I

Continuing the line of thought initiated in Lecture 30, we shall now study properties of the set

$$\mathcal{S} := \{ f \in \mathrm{Hol}(D) \mid f \text{ 1-1},\ f(0)=0,\ f'(0)=1 \}$$

of schlicht (or univalent) functions. What makes it interesting is its compactness in the topology of uniform convergence on compact sets (cf. end of Lecture 1 — henceforth called the normal topology), and the consequent solvability of extremal problems.

## I. Some lemmas

Let $\Omega \subset \mathbb{C}$ be a domain; write $D = D_1$ and "$\mathcal{a}$" for "Area".

**Lemma 1**: $f \in \mathrm{Hol}(\Omega)$ injective $\Longrightarrow \mathcal{a}(f(\Omega)) = \int_\Omega |f'|^2 \, dx\, dy$.

**Proof**: 
$$|f'|^2 = (\mathrm{Re}(f'))^2 + (\mathrm{Im}(f'))^2$$
$$= (u_x)^2 + (v_x)^2 \underset{\text{C-R eqns.}}{=} \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$
$$= \det(J_f).$$
Done by change-of-variable formula. $\square$

**Lemma 2 (Parseval formula)**: If $g(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ and $h(\theta) = \sum_{n\in\mathbb{Z}} d_n e^{in\theta}$ are uniformly convergent on $\mathbb{R}$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,\overline{h(\theta)}\, d\theta = \sum_{n\in\mathbb{Z}} c_n \bar{d}_n ,$$

with RHS absolutely convergent.

**Proof**: Put $S_n(\theta) := \sum_{k=-n}^{n} c_k e^{ik\theta}$, $T_n(\theta) := \sum_{k=-n}^{n} d_k e^{ik\theta}$.

Now $\int_{-\pi}^{\pi} e^{ik\theta} e^{-il\theta}\, d\theta = 2\pi \delta_{kl}$, and our assumption implies that $S_n \overline{T_n} \xrightarrow[\text{unif.}]{n\to\infty} g\overline{h}$; so

$$\int_{-\pi}^{\pi} g\overline{h}\, d\theta = \lim_{n\to\infty} \int_{-\pi}^{\pi} S_n \overline{T_n}\, d\theta = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k \overline{d_k}.$$

If $g=h$, then this is

$$(\infty >) \int_{-\pi}^{\pi} |g|^2\, d\theta = \lim_{n\to\infty} \sum_{k=-n}^{n} |c_k|^2$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} |c_n \overline{d_n}| \underset{\text{Cauchy-Schwarz}}{\leq} \Big(\sum_{n\in\mathbb{Z}} |c_n|^2\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}} |d_n|^2\Big)^{1/2} < \infty. \qquad \square$$

**Remark**// Lemmas 1 & 2 lead (if $\Omega = D$) immediately to
$$a(f(D)) = \pi \sum n |a_n|^2 \quad (\geq \pi \text{ for schlicht}). //$$

**Lemma 3 (Green's formula)**: (a) If $\partial\Omega$ is a $C^1$ Jordan curve, and $f \in C^1(\overline{\Omega})$, then $\int_{\partial\Omega} f\, dz = 2i \int_\Omega \frac{\partial f}{\partial \bar{z}}\, dx\,dy$.
(b) In particular, $\int_{\partial\Omega} \bar{z}\, dz = 2i\, a(\Omega)$.

**Proof**: $\int_{\partial\Omega} P\,dx + Q\,dy = \int_\Omega (Q_x - P_y)\, dx\,dy$ becomes

(with $P = f$, $Q = if$)
$$\int_{\partial\Omega} f \underbrace{(dx + i\,dy)}_{dz} = \int_\Omega i \underbrace{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{2\frac{\partial}{\partial \bar z}} f\, dx\,dy. \qquad \square$$

## II. The area formula

Given $f \in \mathcal{S}$, from $f(0) = 0$ and $f'(0) = 1$ we have

$$f(z) = z + \sum_{n \geq 2} a_n z^n .$$

The Köbe function

$$K(z) := \frac{z}{(1-z)^2} = z \sum_{n \geq 0} \underbrace{\binom{-2}{n}}_{\frac{-2\cdot -3 \cdot \cdots \cdot (-n-1)}{n!} = (-1)^n (n+1)} (-z)^n \overset{m=n+1}{=} \sum_{m \geq 1} m z^m$$

has this form.

__Lemma 4__: $K \in \mathcal{S}$, and $K(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

Proof: Recall $i \frac{1+z}{1-z} : \mathbb{D} \xrightarrow[\cong]{} \mathfrak{h}$, which gives

$$\frac{1+z}{1-z} : \mathbb{D} \xrightarrow[\cong]{} \left\{ \begin{matrix} \text{right} \\ \text{half-plane} \end{matrix} \right\} , \quad \text{hence}$$

(by composing with $(\cdot)^2 : \text{RHP} \xrightarrow{\cong} \mathbb{C} \setminus (-\infty, 0]$) $\quad \left(\frac{1+z}{1-z}\right)^2 : \mathbb{D} \xrightarrow[\cong]{} \mathbb{C} \setminus (-\infty, 0]$. So

$$K = \frac{1}{4} \underbrace{\left( \left(\frac{1+z}{1-z}\right)^2 - 1 \right)}_{= \frac{4z}{(1-z)^2}} : \mathbb{D} \xrightarrow[\cong]{} \mathbb{C} \setminus (-\infty, -\tfrac{1}{4}] ,$$

while $K'(z) = \frac{1-z^2}{(1-z)^4} \Rightarrow K'(0) = 1$; and $K(0) = 0$. $\square$

Write $\hat{D}_r := \hat{\mathbb{C}} \setminus \overline{D_r}$, $\hat{D} := \hat{D}_1$. Related to $\mathcal{S}$ is the set

$$\Sigma := \Big\{ F \in \mathcal{M}er(\hat{D}) \;\Big|\; F \text{ 1-1, } \underbrace{F(\infty)=\infty, \; \underset{\text{holo. at } \infty}{F(z)-z}}_{\text{i.e. } F \text{ has a simple pole at } \infty \text{ and residue 1 there}} \Big\}.$$

Notice that $F \in \Sigma \Rightarrow$

$$F(z) = z + \sum_{n \geq 0} b_n z^{-n}.$$

Examples //

1. $F_0(z) = z + \frac{1}{z}$ is 1-1 on $\hat{D}$ ($\Rightarrow F_0 \in \Sigma$), since

$z + \frac{1}{z} = w \Rightarrow z^2 - wz + 1 = 0$ has $\begin{cases} \text{(a) 2 solutions in } \partial D \\ \text{or} \\ \text{(b) one } \underline{\text{each}} \text{ in } D \;\&\; \hat{D}. \end{cases}$

(a) happens for exactly those $w$ of the form $e^{i\theta} + \frac{1}{e^{i\theta}} = 2\cos\theta$, i.e. $w \in [-2,2]$. So $F(\hat{D}) = \mathbb{C} \setminus [-2,2]$.

2. Given $\lambda \in \mathbb{C} \setminus D$, $b \in \mathbb{C}$, we have

$$F(z) := \lambda^{-1} F_0(\lambda z) + b = z + b + \frac{\lambda^{-2}}{z} \in \Sigma$$

would not be 1-1 if $\lambda \in D$

with $F(\hat{D}) = \mathbb{C} \setminus \text{blob}$ or $\underbrace{\text{slit of length 4}}_{\text{for } |\lambda|=1}$. //

<u>Lemma 5</u>: (i) $f \in \mathcal{S} \Rightarrow \dfrac{1}{f(\frac{1}{z})} \in \Sigma$

(ii) $F \in \Sigma$, $\alpha \notin F(\hat{D}) \Rightarrow \dfrac{1}{F(\frac{1}{z}) - \alpha} \in \mathcal{S}$

(iii) $r \in (1,\infty)$, $F \in \Sigma \Rightarrow \gamma_r(\theta) := F(re^{i\theta})$ is a 'counterclockwise' Jordan curve.

**Proof (of (iii)):** Let $\alpha \in \mathbb{C} \setminus F(\hat{D})$, $g(z) = \frac{1}{F(\frac{1}{z}) - \alpha} \in \mathcal{S}$.

So $\delta_r(\theta) := g\left(\frac{1}{r} e^{i\theta}\right)$ parametrizes $\partial g(D)$ and has counter-clockwise orientation (because $\frac{1}{2\pi i} \oint_{|z| = \frac{1}{r}} \frac{\partial g(z)}{g(z)} = +1$ by argument principle), while $\gamma_r(\theta) = F(re^{i\theta}) = \frac{1}{g(\frac{1}{r} e^{-i\theta})} + \alpha = \delta_r(-\theta)^{-1} + \alpha$.

both reverse orientation

$\square$

## Theorem 1 (Gronwall, 1914)

(i) $F \in \Sigma \Longrightarrow (*) \; \boxed{\sum_{n \geq 1} n|b_n|^2 \leq 1}$, with equality iff $a(\hat{\mathbb{C}} \setminus F(\hat{D})) = 0$

$\Downarrow$

(ii) $|b_1| \leq 1$ ; and $|b_1| = 1 \Rightarrow b_n = 0 \; \forall n \geq 2$.

**Proof of (i):** Let $\gamma_r$ $(r > 1)$ be as in Lemma 5 (iii), and $\Omega_r := \hat{\mathbb{C}} \setminus F(\hat{D}_r)$. (Clearly $\gamma_r = \partial \Omega_r$.) Then

$$2i \, a(\Omega_r) \overset{\text{Lemma 3}}{=} \int_{\gamma_r} \bar{z} \, dz$$

$$= \int_0^{2\pi} \overline{F(re^{i\theta})} \underbrace{F'(re^{i\theta}) \, d(re^{i\theta})}_{dz}$$

$\bar{F} = \bar{z} + \sum_{n \geq 0} \bar{b}_n \bar{z}^{-n}$

$zF' = z - \sum_{n \geq 1} n b_n z^{-n}$

$$= i \int_0^{2\pi} \left\{ re^{-i\theta} + \sum_{n \geq 0} \bar{b}_n r^{-n} e^{in\theta} \right\} \left\{ re^{i\theta} - \sum_{n \geq 1} n b_n r^{-n} e^{-in\theta} \right\} d\theta$$

$$\overset{\text{Parseval}}{=} 2\pi i \left( r^2 - \sum_{n \geq 1} n |b_n| r^{-2n} \right)$$

$$\Rightarrow \sum_{n \geq 1} n|b_n|^2 r^{-2n} = r^2 - \frac{1}{\pi} a(\Omega_r)$$

$$\underset{r \to 1^+}{\Longrightarrow} \sum_{n \geq 1} n|b_n|^2 = 1 - \frac{1}{\pi} a(\hat{\mathbb{C}} \setminus F(\hat{D})) \leq 1.$$

(use Abel's thm. for power series)

$\square$

**Remark**// Any $z + b_0 + \frac{e^{i\varphi}}{z} \in \Sigma$, by Example ② above. //

# III. The $a_2$ theorem

Given $f = z + a_2 z^2 + a_3 z^3 + \dots \in \mathcal{S}$, we have

otherwise f would have more than one 0, violating injectivity

$$f(z) = z f_1(z) \quad \text{with } f_1 \in \mathcal{H}ol(D),\ f_1(0) = 1,\ f_1 \text{ zero-free}$$

$$\underset{\text{D simply connected}}{\Longrightarrow} \exists\, f_2 \in \mathcal{H}ol(D) \text{ with } f_2^2 = f_1,\ f_2(0) = 1.$$

Set $g(z) := z f_2(z^2)$; then $g$ is odd, and

$$(*)\quad \boxed{(g(z))^2 = z^2 (f_2(z^2))^2 = z^2 f_1(z^2) = f(z^2),}$$

i.e. "$g(z) = \sqrt{f(z^2)}$".

<u>Lemma 6</u>: $g \in \mathcal{S}$

Proof: $g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \overset{(*)}{\Longrightarrow} f(z_1^2) = f(z_2^2)$

$$\underset{f \in \mathcal{S}}{\Longrightarrow} z_1 = \pm z_2.$$

Suppose $z_1 \neq z_2$. Then $0 \neq z_1 = -z_2$, and

$$\left.\begin{array}{l} g \text{ odd} \Rightarrow g(z_2) = -g(z_1) \\ \text{while} \\ (*),\ f \in \mathcal{S} \Rightarrow g(z_1) \neq 0 \end{array}\right\} \Rightarrow g(z_2) \neq g(z_1). \ \#$$

$\square$

**Theorem 2 (Bieberbach, 1916)** $f \in \mathcal{S} \Rightarrow |a_2| \leq 2$, with equality iff $f$ is "a rotation of the Köbe function", i.e. $f(z) = e^{-i\alpha} K(e^{i\alpha} z)$ for some $\alpha \in \mathbb{R}$.

**Proof:** With $g$ as above, $g(z) = z + A_3 z^3 + O(z^5)$ (for $z \to 0$)

(since $g \in \mathcal{S}$ and $g$ is odd). Now,

$$(\ast\ast) \implies z^2 + 2A_3 z^4 + O(z^6) = z^2 + a_2 z^4 + O(z^6)$$

$$\implies 2A_3 = a_2 .$$

Set $G(z) := \dfrac{1}{g(\frac{1}{z})} \in \Sigma \qquad (|z|>1)$

$$= \frac{1}{\frac{1}{z} + \frac{A_3}{z^3} + \cdots} = \frac{z}{1 + A_3 z^{-2} + O(z^{-4})}$$

$$= z \underbrace{- A_3}_{\left(\substack{\text{plays role} \\ \text{of } b_1}\right)} z^{-1} + \underset{(\text{for } z \to \infty)}{O(z^{-3})}$$

By Theorem 1, $|A_3| \leq 1$; hence, $|a_2| \leq 2$.

Next, if $|a_2| = 2$, then $|A_3| = 1 \xRightarrow[\text{Thm. 1}]{}$

$$G(z) = z - \frac{A_3}{z} = z - \frac{e^{i\alpha}}{z} \implies g(z) = \frac{1}{G(\frac{1}{z})} = \frac{1}{\frac{1}{z} - e^{i\alpha} z} = \frac{z}{1 - e^{i\alpha} z^2}$$

$$\implies f(z^2) = (g(z))^2 = \frac{z^2}{(1 - e^{i\alpha} z^2)^2} \implies f(z) = \frac{z}{(1 - e^{i\alpha} z)^2} \text{ as claimed.}$$

$\square$

## IV. A first look at distortion

In the next lecture we will prove Köbe's Distortion Theorem (1907), part of which says:

Given $f \in \mathcal{S}$ and $z \in \mathbb{D}$,

(†) $$\boxed{|f(z)| \leq \frac{|z|}{(1-|z|)^2}}.$$

Let $R \in \mathbb{R}_+$ and

$$\mathcal{S}_R := \left\{ f \in \mathcal{H}ol(\mathbb{D}_R) \,\middle|\, f \text{ 1-1},\ f(0)=0,\ f'(0)=1 \right\}.$$

$\boxed{\text{Corollary}}$ $\mathcal{S}_R$ is compact in the normal topology: that is, any sequence has a subsequence normally converging to an element of $\mathcal{S}_R$ itself. (This is stronger than normality.)

**Proof:** Compactness of $\mathcal{S}_R$ will follow from that for $\mathcal{S}$ via the homeomorphism

$$\mathcal{S}_R \to \mathcal{S}$$
$$f(z) \longmapsto \frac{f(Rz)}{R}.$$

According to Montel's theorem, since $\mathcal{S}$ is locally bounded (i.e. on each $\overline{D_r}$, $r<1$), we have that any sequence $\{f_n\} \subset \mathcal{S}$ has a normally convergent subsequence $f_{n_k} \to f$. Clearly $f(0)=0$, $f'(0)=1$ ($\Rightarrow f$ nonconstant).

To see $f$'s injectivity, we must show that $f-\beta$ has at most one zero for any $\beta \in \mathbb{C}$, which follows from the observation ($\forall r<1$, except for values of $|z|$ where $f(z)=\beta$):

[uniform convergence on $\partial D_r$ (which is compact)]

$$(0 \le) \frac{1}{2\pi i} \int_{\partial D_r} \frac{f'}{f-\beta}\, dz = \frac{1}{2\pi i} \oint_{\partial D_r} \frac{f_{n_k}'}{f_{n_k}-\beta} \le 1.$$

So $f \in \mathcal{S}$. $\square$