Lecture 32: Schlicht functions II

Recall (for reference) from Lecture 31 the

$\boxed{a_2 \text{ Theorem (Bieberbach, 1916)}}$ $f \in \mathcal{S} \Rightarrow |a_2| \leq 2$, with equality iff f is "a rotation of the Köbe function", i.e. $f(z) = e^{-i\alpha} K(e^{i\alpha} z)$ for some $\alpha \in \mathbb{R}$.

I. The $\frac{1}{4}$ theorem

Theorem 1 (Köbe, 1907)

(i) $f \in \mathcal{S} \Rightarrow f(\mathbb{D}) \supset \mathbb{D}_{1/4}$

(ii) If (for $f \in \mathcal{S}$) $f(\mathbb{D}) \not\supset \overline{\mathbb{D}_{1/4}}$, then $f(z) = e^{-i\alpha} K(e^{i\alpha} z)$ for some $\alpha \in \mathbb{R}$.

Proof: Let $b \in \mathbb{C} \setminus f(\mathbb{D})$. Since $b \neq 0$,

$$g(z) := \frac{f(z)}{1 - \frac{1}{b} f(z)} \in \mathcal{Hol}(\mathbb{D}), \text{ with } g(0) = 0 \text{ \& } g'(0) = 1.$$

Since $g = \text{FLT} \circ f$, g is 1-1; so $g \in \mathcal{S}$.

Write $g = \dfrac{z + a_2 z^2 + O(z^3)}{1 - \frac{1}{b} z + O(z^2)}$

$$= \left(z + a_2 z^2 + O(z^3)\right)\left(1 + \frac{z}{b} + O(z^2)\right)$$

$$= z + \left(a_2 + \frac{1}{b}\right) z^2 + O(z^3).$$

"a_2 theorem" $\Longrightarrow$ $\left|a_2 + \frac{1}{b}\right| \le 2$ & $|a_2| \le 2$

$\Longrightarrow$ $\dfrac{1}{|b|} = \left|\left(\frac{1}{b} + a_2\right) - a_2\right| \le \left|\frac{1}{b} + a_2\right| + |a_2| \le 4$

$\Longrightarrow$ $f(D) \supset D_{1/4}$ (as b was an arbitrary point not in $f(D)$).

Now, for the extremal case: if f is not a rotation of K, then the a_2 theorem $\Rightarrow$

$\Rightarrow$ $|a_2| = 2 - \epsilon$ for some $\epsilon \in (0, 2]$

$\Rightarrow$ $\dfrac{1}{|b|} \le 2 + (2 - \epsilon) = 4 - \epsilon$

$\Rightarrow$ $|b| \ge \dfrac{1}{4 - \epsilon} > \dfrac{1}{4}$ $\Longrightarrow$ $f(D) \supset \overline{D_{1/4}}$. $\square$

There is a nice application to Smale's conjecture (end of Lect. 30): let $P(z) = z + \sum_{k=2}^{n+1} a_k z^k$ (note $P(0) = 0$, $P'(0) = 1$), $\{z_1, \dots, z_n\}$ = 0's of P', $\{w_1, \dots, w_n\}$ = their images under P, $\sigma(P) := \min_{1 \le j \le n} \dfrac{|w_j|}{|z_j|}$. Conjecture is that $\sigma(P) \le \frac{n}{n+1}$ (known for $n \le 3$).

Theorem 2 (Czerzynski-Kostrzewski/Smale) $\sigma(P) \le 4.$

Proof: [NOTE: read appendix on covering maps first.]

First, $\Omega_2 = \mathbb{C}\setminus\{w_1,\dots,w_n\}$

$$\begin{array}{c} \Omega_2 = \mathbb{C}\setminus\{w_1,\dots,w_n\} \\ P\uparrow \\ \Omega_1 = P^{-1}(\Omega_2) \quad \left(\subset \mathbb{C}\setminus\{z_1,\dots,z_n\}\right) \end{array}$$

is an analytic covering map, since zeroes of P' are omitted. Wolog assume $M := |w_1| \le |w_2| \le \cdots \le |w_n|$; applying the lifting theorem to

$$\begin{tikzcd} & \Omega_1 \arrow[d, "P"] \\ D_M \arrow[ur, dashed, "f"] \arrow[r, hook, "i"', "\text{(inclusion)}"'] & \Omega_2 \end{tikzcd}$$

yields f s.t. $P \circ f = i$. Note that i injective $\Rightarrow$ P injective.

Moreover, we can choose f so that 0 is sent to any point in $P^{-1}(0)$; so choose $f(0)=0$.

Set $f_1(z) := \dfrac{f(zM)}{M} \in \mathcal{H}ol(D)$. We have

$$1 = i'(0) = P'(f(0))\cdot f'(0) = \cancel{P'(0)}^{1}\cdot f'(0) = f'(0)$$

$\Rightarrow f_1'(0) = f'(0) = 1, \quad f_1(0) = 0$

$\Rightarrow f_1 \in \mathcal{S}$.

Since $z_j \notin \Omega_1$, $f(D_M)$ contains no z_j's

$\Rightarrow f_1(D) \left(= \frac{1}{M} f(D_M)\right)$ contains no $\frac{z_j}{M}$.

By Köbe $\frac{1}{4}$, $f_1(D) \supset D_{1/4}$, and so each $\left|\frac{z_j}{M}\right| \ge \frac{1}{4}$.

In particular, $|z_1| \ge \dfrac{M}{4} = \dfrac{|w_1|}{4} \Rightarrow \min \dfrac{|w_j|}{|z_j|} \le \dfrac{|w_1|}{|z_1|} \le 4$. $\square$

II. Distortion

As usual $D = D_1$.

Lemma 1: Given $\begin{cases} \gamma : [0,1] \to D & \text{path from } 0 \text{ to } z_0 \\ g \in C^0(0, |z_0|), & g \geq 0 \end{cases}$

Then $\displaystyle\int_\gamma g(|z|) \underbrace{|dz|}_{\text{i.e. } ds} \geq \int_0^{|z_0|} g(r)\, dr$.

Proof: Although this isn't C^0, it's enough to prove this for $g = \chi_{(a,b)}$ (characteristic function), with $(a,b) \subset (0, |z_0|)$.

We have

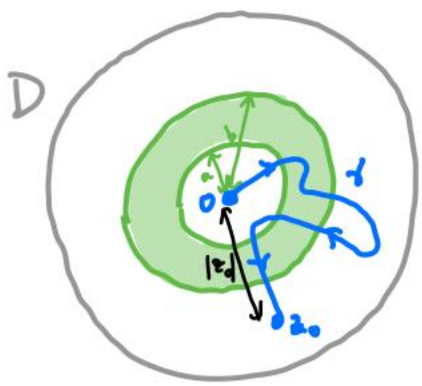
$$\int_\gamma g(|z|)|dz| = \int_{\substack{\gamma_{a,b} := \gamma \cap \{|z| \in (a,b)\} \\ = \gamma(E),\ E \subset (0,1)}} |dz| = \int_E |\gamma'(t)|\, dt \qquad (\dagger)$$

← need not be connected

Let $t_2 := \min\{t \in (0,1] \mid |\gamma(t)| = b\}$

$t_1 := \max\{t \in [0, t_2) \mid |\gamma(t)| = a\}$.

Then $[t_1, t_2] \subset E \implies$



$$\begin{aligned}
(\dagger) &\geq \int_{t_1}^{t_2} |\gamma'(t)|\, dt \\
&\geq \left| \int_{t_1}^{t_2} \gamma'(t)\, dt \right| \\
&= |\gamma(t_2) - \gamma(t_1)| \\
&\geq |\gamma(t_2)| - |\gamma(t_1)| \\
&= b - a \\
&= \int_0^{|z_0|} g(r)\, dr.
\end{aligned}$$

$\square$

Recall the Köbe function

$$K(z) := z/(1-z)^2$$

with $$K'(z) = (1+z)/(1-z)^3.$$

We have

$$\sup_{|z|=r} |K(z)| = \frac{r}{(1-r)^2} = K(r), \quad \inf_{|z|=r} |K(z)| = \frac{r}{(1+r)^2} = K(-r),$$

$$\sup_{|z|=r} |K'(z)| = \frac{1+r}{(1-r)^3} = K'(r), \quad \inf_{|z|=r} |K'(z)| = \frac{1-r}{(1+r)^3} = K'(-r).$$

$\boxed{\text{Theorem 3 (Köbe, 1907; Bieberbach, 1916)}}$ Given $f \in \mathcal{S}$, $r \in (0,1)$:

(a) We have (i) $\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$

(ii) $\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$ $\quad \forall z \in \partial D_r$.

(b) If for some $z \in \partial D_r$, one of the four inequalities in (a) is an equality, then f is a rotation of K.

Proof:

(a) [RHS of (i) & (ii)]: Write $\tau_\alpha(z) = \frac{z+\alpha}{1+\bar{\alpha}z}$, $\alpha \in D$.

Set $g(z) := \frac{f(\tau_\alpha(z)) - f(\alpha)}{\underbrace{(f\circ\tau_\alpha)'(0)}_{= f'(\alpha)\tau_\alpha'(0)}} \in \mathcal{S}$ (g is 1-1 because of the form FLT∘f∘FLT).

By the "a_2 Theorem", $|g''(0)| = |2a_2| \le 4$.

Now $\tau_\alpha'(z) = \dfrac{1-|\alpha|^2}{(1+\bar\alpha z)^2}$, $\dfrac{\tau_\alpha''}{\tau_\alpha'}(z) = \dfrac{-2\bar\alpha}{1+\bar\alpha z}$

$\Rightarrow \tau_\alpha'(0) = 1-|\alpha|^2$, $\dfrac{\tau_\alpha''}{\tau_\alpha'}(0) = -2\bar\alpha$, $\tau_\alpha''(0) = -2\bar\alpha\,(1-|\alpha|^2)$,

while $g'(z) = f'(\tau_\alpha(z))\,\tau_\alpha'(z) \big/ f'(\alpha)\,\tau_\alpha'(0)$

and $g''(z) = \left\{ f''(\tau_\alpha(z))(\tau_\alpha'(z))^2 + f'(\tau_\alpha(z))\,\tau_\alpha''(z) \right\} \big/ f'(\alpha)\tau_\alpha'(0)$.

So

$$4 \ge |g''(0)| \overset{\tau_\alpha(0)=\alpha}{=} \left| \frac{f''(\alpha)\,(\tau_\alpha'(0))^2 + f'(\alpha)\,\tau_\alpha''(0)}{f'(\alpha)\,\tau_\alpha'(0)} \right|$$

$$= \left| \frac{f''}{f'}(\alpha)\cdot \tau_\alpha'(0) + \frac{\tau_\alpha''}{\tau_\alpha'}(0) \right|$$

$$= \left| \frac{f''(\alpha)}{f'(\alpha)}(1-|\alpha|^2) - 2\bar\alpha \right|.$$

Changing α to z, and multiplying by $\dfrac{|z|}{1-|z|^2}$, we get

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \le \frac{4|z|}{1-|z|^2},$$

from which we deduce

$$(*) \quad \frac{-4|z|+2|z|^2}{1-|z|^2} \le \mathrm{Re}\left(\frac{z f''(z)}{f'(z)}\right) \le \frac{4|z|+2|z|^2}{1-|z|^2}.$$

But then, since

$$K''(z) = \frac{4+2z}{(1-z)^4}, \quad \frac{K''}{K'}(z) = \frac{4+2z}{(1-z^2)},$$

for $|z|=r$ we have

(**) $$-r\cdot\frac{K''}{K'}(-r) \le \underbrace{\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) \le r\cdot\frac{K''}{K'}(r)}_{\downarrow \text{ take } z=r}$$

$$\frac{d}{dr}\mathrm{Re}\left(\log(f'(r))\right) = \mathrm{Re}\left(\frac{f''(r)}{f'(r)}\right) \le \frac{K''(r)}{K'(r)} = \frac{d}{dr}\log K'(r).$$

Since the logs are 0 at $r=0$ (why?), integrating $\int_0^r$ yields

$$\mathrm{Re}\left(\log(f'(r))\right) \le \log K'(r)$$

$$\Longrightarrow \quad |f'(r)| \le K'(r). \qquad (\dagger\dagger)$$

Integrating once more (using that $f(0)=K(0)=0$) yields

$$|f(r)| \le \int_0^r |f'(t)|\,dt \le K(r). \qquad (\dagger\dagger\dagger)$$

Given θ_0, let $f_1(z) := e^{-i\theta_0} f(ze^{i\theta})$ $(\in \mathcal{S})$; then

$$\left.\begin{aligned} |f(re^{i\theta_0})| &= |f_1(r)| \le K(r) \\ |f'(re^{i\theta_0})| &= |f_1'(r)| \le K'(r) \end{aligned}\right\} \Rightarrow \text{done w/RHS of (i)\&(ii).}$$

by $(\dagger\dagger)$&$(\dagger\dagger\dagger)$

(a) [LHS of (i)&(ii)]: The left-hand part of (**) gives

$$\frac{d}{dr}\log K'(-r) = -\frac{K''}{K'}(-r) \le \mathrm{Re}\,\frac{f''}{f'}(r) = \frac{d}{dr}\mathrm{Re}\log f'(r) \quad \Big\rangle \int_0^r$$

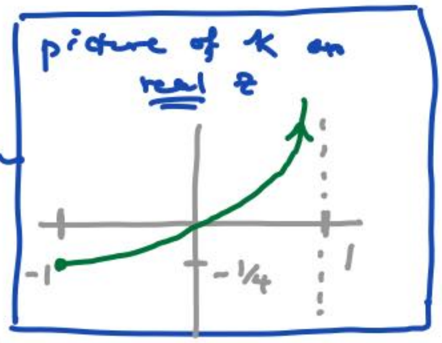
$$\log K'(-r) \le \mathrm{Re}\log f'(r) = \log|f'(r)| \quad \Big\rangle \exp$$

$$K'(-r) \le |f'(r)|, \quad \text{same for } f_1,$$

hence $K'(-r) \le |f'(re^{i\theta_0})| \Rightarrow$ LHS of (a)(i).

Next, take $z_0 = r_0 e^{i\theta_0} \in D$; then

$$|K(-r_0)| \le |K(-1)| = \frac{1}{4}.$$



CASE 1 : $|f(z_0)| \ge \frac{1}{4}$: then
$|f(z_0)| \ge |K(-r_0)|$ ($\Rightarrow$ LHS of (a)(ii))

CASE 2 : $|f(z_0)| < \frac{1}{4}$: then the radial segment $[0, f(z_0)]$ belongs to $D_{1/4}$, hence (by Köbe $\frac{1}{4}$ theorem) to $f(D)$.
Set $\gamma := f^{-1}([0, f(z_0)])$ (= path from 0 to z_0) $\subset D$.
Then
$$\begin{aligned} |f(z_0)| = \int_{[0,f(z_0)]} |dw| &= \int_\gamma |f'(z)|\,|dz| \\ &\ge \int_\gamma K'(-|z|)\,|dz| \\ \text{Lemma 1} \rightsquigarrow &\ge \int_0^{|z_0|} K'(-R)\,dR \\ &= -K(-|z_0|) \Rightarrow \text{LHS (a)(ii).} \end{aligned}$$

(b) : Apply (*) with $z = r$:

$$\frac{2r-4}{1-r^2} \le \operatorname{Re} \frac{f''(r)}{f'(r)} \le \frac{2r+4}{1-r^2} \qquad (\#)$$

i.e. $\frac{d}{dr} \log K'(-r) \le \frac{d}{dr} \log |f'(r)| \le \frac{d}{dr} \log K'(r)$.

If $|f'(r_0)| = K'(r_0)$ for some $r_0 \in (0,1)$, then clearly RHS (#) is an equality for all $r \in (0, r_0)$. Taking $r \to 0$, we get $2\operatorname{Re}(a_2) = 4 \overset{|a_2| \le 2}{\Rightarrow} |a_2| = 2 \overset{a_2 \text{ Thm.}}{\Longrightarrow} f =$ rotation of K.

In fact, $\operatorname{Re}(a_2) = |a_2| = 2 \Rightarrow a_2 = 2 \Rightarrow f = K$.

If $|f(r_0)| = K(r_0)$ then $|f'(r)| = K'(r)$ $\forall r \in (0, r_0)$,

which again (by same argument) $\Rightarrow f = K$.

If $|f'(r_0)| = K(-r_0)$ or if $|f(r_0)| = -K(-r_0)$,

an analogous argument $\Rightarrow f(z) = -K(-z)$. $\square$

Appendix: Covering spaces

Let $\Omega_1 \xrightarrow{F} \Omega_2$ be a continuous mapping of topological spaces. F is called a covering map if it is surjective and a local homeomorphism. (More precisely, one should require that there is a "discrete space" $\mathcal{D}$ and for each $p \in \Omega_2$ a nbhd. $(p \in) U \subset \Omega_2$ s.t. $F^{-1}(U) = \coprod_{\delta \in \mathcal{D}} V_\delta$ with $F|_{V_\delta} : V_\delta \to U$ local homeo.) When F is an analytic map of regions (or Riemann surfaces), this will mean that a sufficiently small disk $\mathbb{D}$ about each point in Ω_2 has preimage equal to a (nonempty) disjoint union of disk-like blobs in Ω_1 each of which F maps 1-1 conformally onto $\mathbb{D}$. Equivalent conditions are that the derivative F' be everywhere nonzero ("F étale"), or that there be no ramification points (where $z \overset{F}{\mapsto} z^{k>1}$ locally); this is enough because of the inverse mapping theorem.

Lifting Theorem Given Ω_0 = simply connected region & $g : \Omega_0 \to \Omega_2$ holomorphic, there exists a (holomorphic) "lifting" map $G : \Omega_0 \to \Omega_1$ such that $F \circ G = g$.

Moreover, given $z_0 \in \Omega_0$ and <u>any</u> $w_0 \in F^{-1}(g(z_0))$, we may arrange that $G(z_0) = w_0$.

$$\begin{tikzcd} & \Omega_1 \arrow[d, "F"] \\ \Omega_0 \arrow[ur, dotted, "\exists G"] \arrow[r, "g"'] & \Omega_2 \end{tikzcd}$$

Sketch: Taking a sufficiently small ball B about $z_0 \in \Omega_0$ and composing g with a local branch of F^{-1} on $g(B)$, gives a germ G_0 at z_0. Any path on Ω_2 lifts (uniquely, after fixing the initial point's lift) to a path on Ω_1 — which may <u>not</u> be closed even if the one on Ω_2 <u>is</u>.

In particular, taking a path in Ω_0 from z_0, we can cover it with balls sufficiently small that their g-images have homeomorphic preimages under F covering the lifted path. In this way one gets an analytic continuation of G_0 along all paths in Ω_0, and since Ω_0 is simply connected, we are done by the monodromy theorem. $\square$