Lecture 33: Green's functions on Riemann surfaces I

I. Riemann surfaces and universal covers

$\boxed{\text{Definition}}$ (i) Let M be a connected Hausdorff space, $\mathcal{U} = \{U_\alpha \mid U_\alpha \subset M\}$ an open cover of M, $z_\alpha : U_\alpha \to \mathbb{C}$ continuous 1-1 maps such that the homeomorphisms $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\cong} z_\beta(U_\alpha \cap U_\beta)$ are (bi)holomorphic. Then $\mathcal{A} = \{(z_\alpha, U_\alpha)\}$ is a conformal atlas, $(M, \mathcal{A})$ a Riemann surface (usually denoted "M").

(ii) A function f on M is holomorphic $\overset{\text{def}}{\iff} \{f \circ z_\alpha^{-1}\}$ are holomorphic.

(iii) A map $F : M \to M'$ of RS's is holomorphic $\overset{\text{def}}{\iff} \{{}'z_\beta \circ F \circ z_\alpha^{-1}\}$ are holomorphic.

This turns out to be equivalent to the "multivalued analytic function element" approach, simply because one can always produce a nontrivial pair of holomorphic maps $M \to \mathbb{P}^1$ such that $M \underset{(f,g)}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$ is an immersion (with finitely many transverse crossings in the image).

Every RS M has a universal cover

$$\tilde{M} \xrightarrow{\varphi} M,$$

which is a simply connected covering space.† One first constructs this topologically, then layers on the holomorphic structure. One may view $M \cong {}_{\Gamma}\backslash\tilde{M}$ as the quotient of $\tilde{M}$ by a group of "deck transformations" (permuting branches of $\tilde{M}$ over M), acting properly discontinuously†† (so that the quotient is Hausdorff). This makes sense because the preimage of a sufficiently small neighborhood of M is a tower of neighborhoods on $\tilde{M}$, and Γ acts by holomorphic automorphisms of $\tilde{M}$.

Suppose now $\tilde{M} = \mathbb{C}$. Recall that $\mathrm{Aut}(\mathbb{C})$ consists of transformations of the form $z \mapsto \alpha z + \beta$. This has 3 types of "properly discontinuous subgroup":

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†† that is, every $x \in \tilde{M}$ has a nbhd. U s.t. $gU \cap U \neq \emptyset \overset{(g\in\Gamma)}{\Longrightarrow} g = id$. To ensure the quotient is Hausdorff, we actually need a little more: for $x, x' \in \tilde{M}$ not in the same Γ-orbit, $\exists$ nbhds. U, U' s.t. $gU \cap U' = \emptyset$ $(\forall g \in \Gamma)$.

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† recall this means (roughly) that φ is surjective (onto) & a <u>local</u> <u>homeomorphism</u> — no ramification allowed!!

- trivial sgp. $\{0\} \implies M = \mathbb{C}$
- $\mathbb{Z}\langle\lambda\rangle$ ($\lambda \in \mathbb{C}^*$ fixed) $\implies M = {}_{\lambda\mathbb{Z}}\backslash^{\mathbb{C}} \cong {}_{2\pi i\mathbb{Z}}\backslash^{\mathbb{C}} \xrightarrow[\exp]{\cong} \mathbb{C}^*$
- $\underbrace{\mathbb{Z}\langle\lambda_1,\lambda_2\rangle}_{\text{"}\Lambda\text{"}}$ (λ_1, λ_2 indep/$\mathbb{R}$) $\implies M = {}_{\Lambda}\backslash^{\mathbb{C}} \cong$ ell. curve E_Λ

Next suppose $\widetilde{M} = \hat{\mathbb{C}}$ (i.e. $\mathbb{P}^1$). We had $\mathrm{Aut}\,\hat{\mathbb{C}} \cong PSL_2(\mathbb{C})$, but there are no "properly discontinuous subgroups" as all of these automorphisms have fixed points (blame $\hat{\mathbb{C}}$'s compactness). So the only option here is $M = \hat{\mathbb{C}}$.

The Uniformization Theorem, which we'll prove next week, essentially says that $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$, and E_Λ is the complete list of RS's whose universal cover is not D (equivalently $\mathfrak{h}$). You may be familiar with the topology result that every compact orientable real 2-manifold, hence every compact RS, is a "sphere with handles attached":

Uniformization asserts that any compact RS of genus $g \geq 2$

(or for that matter $E\setminus$pt. or $\hat{\mathbb{C}}\setminus$3 pts.) is $\Gamma\backslash D$ (or $\Gamma\backslash\mathfrak{h}$) for some $\Gamma \subset \mathrm{Aut}(D) \cong \mathrm{Aut}(\mathfrak{h}) \cong PSL_2(\mathbb{R})$. The Poincaré metric on D induces through this a metric of constant negative curvature on all but the 4 "exceptional" types of RS above!

Though others, including Köbe and Poincaré, made the proof more rigorous and natural, the idea of uniformization (and its proof) were due to Klein in 1882. The way he thought of it was that every RS M could be parametrized by a single complex variable defined on a subset of $\hat{\mathbb{C}}$. That means you can write (say) all meromorphic functions on M in terms of this variable. We've already seen how powerful this is in the case of elliptic functions (defining functions on E) and modular functions (defining functions on modular curves like $\Gamma(N)\backslash\mathfrak{h}$).

On Friday we'll prove some technical results about Green's functions on RSs, which will be needed in the proof of uniformization. (Another tool will be the Distortion Theorem of Lect. 32.) In the next short section, I set up (some of) the language.

II. Green's functions

Let M be a RS, and let $V \subset \underline{\mathcal{H}}(M)$ be a subset ("family"). (This time we will allow our subharmonic functions to tend to $-\infty$ at isolated points; but we will otherwise still take them to be continuous.) Note that the notion of subharmonicity is preserved under "precomposition" with holomorphic functions (viz., $v \circ f$); otherwise the notion wouldn't make sense on RSs.

Definition V is Perron $\Longleftrightarrow \begin{cases} \text{(i) } v_1, v_2 \in V \Rightarrow \max(v_1, v_2) \in V \\ \text{(ii) } v \in V,\ \Delta \subset M \text{ Jordan region} \\ \qquad \Rightarrow \hat{v}_\Delta \in V. \end{cases}$

local harmonic majorization

Proposition $\underline{V \text{ Perron}} \Rightarrow \underline{u := \sup\limits_{v \in V} v}$ is either identically $+\infty$ or belongs to $\mathcal{H}(M)$.

pointwise

Proof: We did this for plane regions†; generalizes easy to RS. $\square$

Definition Let $p_0 \in M$, z a local coordinate about p_0,

$V_{p_0} :=$ family of $v \in \underline{\mathcal{H}}(M \setminus \{p_0\})$ s.t. $\begin{cases} v \equiv 0 \text{ outside a compact set,} \\ \text{and} \\ \overline{\lim\limits_{p \to p_0}} \{v(p) + \log|z(p)|\} < \infty. \end{cases}$

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† look at Step 2 in the proof of Dirichlet (lecture 10)

If $\sup_{v \in V_{p_0}} v$ is finite, we say M has a Green's function at p_0, denoted $g(p, p_0)$.

Remark// (i) If a GF $\exists$ at p_0, the range of $z(p)$ contains some D_{r_0}. Set $v_0(p) := \log r_0 - \log |z(p)|$ where $|z(p)| \leq r_0$ and $v_0(p) = 0$ elsewhere $\Rightarrow v_0 \in V_{p_0}$
$\Rightarrow g(p, p_0) \geq v_0(p) \Rightarrow \lim_{p \to p_0} g(p, p_0) = +\infty$
($\Rightarrow$ nonconstant)

(ii) M compact $\Rightarrow \nexists$ Green's fcn. at any pt.
(otherwise, $g(p, p_0)$ would have a minimum, hence be constant)

(iii) $g(p, p_0) > 0$ (since $0 \in V_{p_0}$, "≥ 0" is clear; and M open $\Rightarrow$ g can't attain its minimum). //

III. Schlicht functions, concl.

There are a couple more interesting things to tell...

(A) MEANS

$\boxed{\text{Theorem 1 (Littlewood, 1925)}}$ $f \in \mathcal{S}$, $r \in (0,1) \Longrightarrow$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\,d\theta \le \frac{r}{1-r}.$$

To prove this, we require a

Lemma: Given $g \in \mathcal{H}ol(\mathbb{D})$, set (for $r \in (0,1)$)

$$I(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^2\,d\theta.$$

Then $r\,I'(r) = \frac{4}{2\pi}\int_{D_r} |g'(z)|^2\,dA.$

Proof:

$$g(z) = \sum_{n\ge 0} b_n z^n$$

$$g'(z) = \sum_{n\ge 1} n b_n z^{n-1} = \sum_{m\ge 0} (m+1) b_{m+1} z^m.$$

By Parseval,

$$I(r) = \sum_{n\ge 0} |b_n|^2 r^{2n}, \quad r\,I'(r) = \sum_{n\ge 1} 2n|b_n|^2 r^{2n},$$

and

$$\frac{1}{2\pi}\int_{D_r} |g'(z)|^2\,dA = \frac{1}{2\pi}\int_0^r \rho\,d\rho \int_0^{2\pi} |g'(\rho e^{i\theta})|^2\,d\theta$$

$$= \int_0^r \rho \sum_{n\ge 1} n^2 |b_n|^2 \rho^{2n-2}\,d\rho$$

$$= \sum_{n\ge 1} n^2 |b_n|^2 \int_0^r \rho^{2n-1}\,d\rho$$

$$= \sum_{n\ge 1} \tfrac{1}{2} n |b_n|^2 r^{2n} = \tfrac{1}{4} r\,I'(r). \quad \square$$

Proof of Thm. 1 : Let $f \in \mathcal{S}$, $g(z) = \sqrt{f(z^2)}$ $(\in \mathcal{S})$.

We have
$$I(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^2\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(r^2 e^{2i\theta})|\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(r^2 e^{i\theta})|\, d\theta .$$

Now, g is 1-to-1; by the lemma,
$$r I'(r) = \frac{4}{2\pi}\int_{D_r} |g'|^2\, dA = \frac{4}{2\pi}\mathcal{A}(g(D_r))$$
$$\leq \frac{4}{2\pi}\pi \|g\|_{D_r}^2 = 2\|f\|_{D_{r^2}}$$
$$\leq 2\|K\|_{D_{r^2}} = \frac{2r^2}{(1-r^2)^2} .$$

So
$$\left\{\begin{array}{l} I'(r) \leq \dfrac{2r}{(1-r^2)^2} = \dfrac{d}{dr}\left(\dfrac{1}{1-r^2}\right) \\ I(0) = |g(0)|^2 = 0 , \quad \dfrac{1}{1-r^2}\Big|_0 = 1 \end{array}\right\}$$

$$\Rightarrow I(r) \leq \frac{1}{1-r^2} - 1 = \frac{r^2}{1-r^2} .$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta = I(\sqrt{r}) \leq \frac{r}{1-r} .$$

$\square$

Remark //
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r}{|1-re^{i\theta}|^2}\, d\theta$$

expand $\frac{1}{1-re^{i\theta}} \cdot \frac{1}{1-re^{-i\theta}}$ $\rightsquigarrow$
$$= r\sum_{n\geq 0} r^{2n} = \frac{r}{1-r^2} = \frac{1}{1+r}\cdot\frac{r}{1-r} \underset{(r\to 1^-)}{\sim} \frac{1}{2}\cdot\frac{r}{1-r} .$$

So Littlewood's estimate is asymptotically too high by a factor of 2, in light of the following result:

$\boxed{\text{Theorem (Baernstein, 1974)}}$ $f \in \mathcal{S}$, $p \in \mathbb{R}$, $r \in (0,1) \Longrightarrow$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |k(re^{i\theta})|^p \, d\theta.$$

Equality for any $r, p \Rightarrow f =$ rotation of k. //

(B) COEFFICIENTS

As usual, we take $f = z + \sum_{n \ge 2} a_n z^n \in \mathcal{S}$.

$\boxed{\text{Theorem 2 (Littlewood, 1925)}}$ $|a_n| \le e \cdot n, \quad n \ge 2$.

Proof: $r^n a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta} \, d\theta$ (by Cauchy)

$$\Rightarrow \quad r^n |a_n| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})| \, d\theta \underset{\text{Thm. 1}}{\le} \frac{r}{1-r}.$$

Taking $r = 1 - \frac{1}{n}$, $\left(1-\frac{1}{n}\right)^n |a_n| \le n\left(1-\frac{1}{n}\right)$.

So $|a_n| \le n\left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} n < en$

take log + do L'Hôpital:

$$\lim_{x\to\infty} \frac{\log\left(\frac{x}{x-1}\right)}{\frac{1}{x-1}} = \lim_{x\to\infty} \frac{\frac{-1}{x(x-1)}}{\frac{-1}{(x-1)^2}} = \lim_{x\to\infty} \frac{x-1}{x} = 1$$

(and $\frac{x-1}{x} < 1$)

$\square$

We won't prove the Bieberbach Conjecture / de Branges Theorem (i.e. the statement that $|a_n| \leq n$, and that equality for some n $\Rightarrow$ f is a rotation of K), but here is an easy special case:

$\boxed{\text{Theorem 3 (Dieudonne, 1931)}}$ Given $f \in \mathcal{S}$ with all $a_n \in \mathbb{R}$, we have $|a_n| \leq n$ $\forall n \geq 2$.

Proof: First, observe that $f(D \cap h) \subset h$ or $-h$:

otherwise, since the image is connected, $\exists z_0 \in D \cap h$ s.t. $f(z_0) \in \mathbb{R}$; and then $\{a_n\} \subset \mathbb{R}$ $\Rightarrow$ $f(z_0) = f(\overline{z_0})$ $\Rightarrow$ f not 1-1 $\times$

In fact, it's clear that $f(D \cap h) \subset h$: just look at the form of $f \sim f(z) = z + O(z^2)$.

Now let $V := \mathrm{Im}(f)$. Then $V\big|_{D \cap h} > 0$, while

$$V(re^{i\theta}) = \sum_{n \geq 1} a_n r^n \sin(n\theta)\ ; \quad \text{so for } n \geq 1$$

$$r^n a_n = \frac{2}{\pi} \int_0^\pi V(re^{i\theta}) \underbrace{\sin(n\theta)}_{|\cdot| \leq n \sin\theta} \, d\theta$$

$$\Rightarrow r^n |a_n| \leq \frac{2}{\pi} \int_0^\pi V(re^{i\theta})\, n \sin\theta \, d\theta$$

$$= n \cancel{a_1}^{1} r = nr.$$

Letting $r \to 1^-$, we get $|a_n| \le n$. $\square$

Here is one more exotic variant. Recall $E \subset \mathbb{C}$ is <u>starlike</u> with resp. to 0 $\Leftrightarrow$ $\forall z \in E$, $t \in [0,1]$ we have $tz \in E$.

Set $\mathcal{S}^* := \{ f \in \mathcal{S} \mid f(\mathbb{D}) \text{ is starlike w.r.t. } 0 \}$.

$\boxed{\text{Theorem (Nevanlinna, 1920)}}$ $f \in \mathcal{S}^* \Rightarrow |a_n| \le n \quad (n \ge 2)$,

with the usual statement on "equality for some $n \Rightarrow f =$ rot. of K."