Lecture 34: Green's functions on Riemann surfaces II

Let M be a RS, and let $V \subset \underline{\mathcal{H}}(M)$ be a subset ("family"). Recall:

$\boxed{\text{Definition}}$ V is Perron $\iff \begin{cases} (i)\ v_1, v_2 \in V \Rightarrow \max(v_1, v_2) \in V \\ (ii)\ v \in V,\ \Delta \subset M \text{ Jordan region} \\ \quad \Rightarrow \hat{v}_\Delta \in V. \end{cases}$

$\boxed{\text{Proposition 1}}$ V Perron $\Rightarrow$ $u := \sup_{v \in V} v$ is either identically $+\infty$ or belongs to $\mathcal{H}(M)$.

$\boxed{\text{Definition}}$ Let $p_0 \in M$, z a local coordinate about p_0,

$V_{p_0} :=$ family of $v \in \underline{\mathcal{H}}(M \setminus \{p_0\})$ s.t. $\begin{cases} v \equiv 0 \text{ outside a compact set,} \\ \text{and} \\ \overline{\lim}_{p \to p_0} \{v(p) + \log|z(p)|\} < \infty. \end{cases}$

If $\sup_{v \in V_{p_0}} v$ is finite, we say M has a Green's function at p_0, denoted $g(p, p_0)$.

If M is compact, no GFs exist.

If $g(p, p_0)$ exists, it is $\underline{\underline{\geq 0}}$ and limits to $\underline{\underline{+\infty}}$ at p_0.

Now let $K \subset M$ be compact w/nonempty interior and $M \setminus K$ connected,

$V_K \subset \underline{H}(M \setminus K)$ the (Perron) family s.t. $\begin{cases} v \in V_K \text{ is } \leq 1 \ (\text{on } M \setminus K) \\ v \in V_K \Rightarrow \overline{\lim}\, v(p_n) \leq 0 \\ \quad \text{when "} p_n \to \infty \text{"} \end{cases}$

$\underbrace{}_{\text{i.e. eventually leaving any compact subset of } M}$

(sup)

Clearly this has $0 \leq u_K \leq 1$. Looking at the particular elements of V_K defined by

$$v(p) := \begin{cases} \log_2\left(\dfrac{2\delta}{|z(p) - z_0|}\right), & p \in \text{annulus} \\ 0, & \text{otherwise} \end{cases}$$

we see $u_K \not\equiv 0$, hence $u_K > 0$ (by the maximum principle for harmonic).
Similarly if $u_K \not\equiv 1$ then $u_K < 1$.

$\boxed{\text{Definition}}$ (a) If $u_K \not\equiv 1$ then call it the harmonic measure of K (i.e. this exists).

(b) Say the maximum principle is valid on $M \setminus K$ if, for every bounded-above $u \in \mathcal{H}(M\setminus K)$,
$\overline{\lim_{p \to K}}\, u(p) \le 0$ or β $\implies$ $u \le 0$ or β on $M \setminus K$.

$\boxed{\text{Proposition 2}}$ The following are equivalent:

Main pt.: These are properties of the RS M, not of p_0 or K.

(i) Green's functions exist ($\forall p_0 \in M$)

(ii) Harmonic measures exist ($\forall K \subset M$)

(iii) The maximum principle is NOT valid ($\forall M \setminus K$).

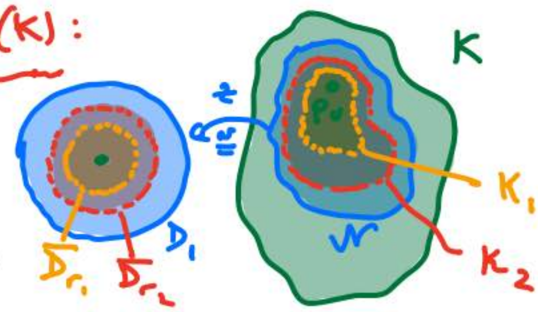
Moreover, $\exists$ of GF [resp. HM] for one p_0 [resp. one K] ensures their existence for all p_0 [resp. K].

Proof:

$\boxed{\text{STEP 1}}$ $(i)_{p_0} \implies (iii)_K$ if $p_0 \in K$:

$-g(p,p_0)$ ($\to -\infty$ at p_0) has a maximum μ on K, and is bounded above by 0 on $M \setminus K$. The maximum principle for $M \setminus K \implies$ $-g \le \mu \implies \mu =$ maximum value for $-g$ on all of $M \setminus p_0$, assumed at some point of $K \setminus p_0$ $\xRightarrow{\text{max. princ. for } M\setminus p_0}$ g constant. $\times$

$\boxed{\text{STEP 2}}$ $(ii)_K \Rightarrow (i)_{p_0}$ if $p_0 \in \mathrm{int}(K)$:



If $u_{K_1} \nexists$, then $\sup\limits_{v \in V_{K_1}} \equiv 1 \overset{V_{K_1} \subset V_K}{\Longrightarrow}$ $\sup\limits_{v \in V_K} \equiv 1 \Rightarrow u_K \nexists$. So assuming now that u_K exists, we have u_{K_1} too.

Now given $v \in V_{p_0}$, we consider $v^+ := \max(v, 0) \in V_{p_0}$. Notice that $v^+(p) \le \left(\max\limits_{\partial K_1} v^+\right) \cdot u_{K_1}(p)$ holds on ∂K_1 and also near the "ideal boundary" ($p \to \infty$) [since $v^+ \in V_{p_0}$ vanishes outside a compact set]. So then this "$\le$" must hold outside K_1 by the usual maximum principle (on $M \setminus K_1$), and we have in particular

$$(*) \qquad \max_{\partial K_2} v^+ \le \left(\max_{\partial K_1} v^+\right)\left(\max_{\partial K_2} u_{K_1}\right).$$

Next look at $v^+(p) + (1+\epsilon)\log|z(p)|$ on K_2, which $\to -\infty$ as $p \to p_0$ (since $\epsilon > 0$). It follows that

$$(**) \qquad \max_{\substack{\partial K_1 \\ \text{or } K_1}} v^+ + (1+\epsilon)\log r_1 \le \max_{\substack{\partial K_2 \\ \text{or } K_2}} v^+ + (1+\epsilon)\log r_2 .$$

Now $(**) + (*) \Longrightarrow$

$$\Rightarrow \max_{\partial K_1} v^+ \le \max_{\partial K_2} v^+ + (1+\epsilon)\log\left(\frac{r_2}{r_1}\right) \le \left(\max_{\partial K_1} v^+\right)\left(\max_{\partial K_2} u_{K_1}\right) + (1+\epsilon)\log\left(\frac{r_2}{r_1}\right)$$

$$\Rightarrow \max_{\partial K_1} v^+ \le \frac{(1+\epsilon)\log(r_2/r_1)}{1 - \underbrace{\max\limits_{\partial K_2} u_{K_1}}_{<1}}$$

$\Rightarrow$ v^+, hence v, is uniformly bounded above on ∂K_1

$\Rightarrow$ $g(p, p_0)$ exists.

$\boxed{\text{STEP 3}}$ $(iii)_K \Rightarrow (ii)_{K'}$ $\forall K, K'$: (This will finish the proof.)

We show $\cancel{(ii)}_{K'} \Rightarrow \cancel{(iii)}_K$; so assume $u_{K'} \nexists$, i.e.
$\sup_{v \in V_{K'}} v \equiv 1$.

First assume $K' \subset K$. Let $v \in V_{K'}$, and $u \in \mathcal{H}(M \setminus K)$ with $u \le 1$ and $\overline{\lim}_{p \to K} u(p) \le 0$. Then $v + u \le 1$ as $p \to \infty$, $p \to K$ $\Longrightarrow$ $v + u \le 1$ on $M \setminus K$ (max. principle). So taking v arbitrarily close to 1, we find $u \le 0$. Hence the maximum principle is valid on $M \setminus K$.

If $K' \not\subset K$, choose K'' s.t. $K \cup K' \subset \mathrm{int}(K'')$. The assumed nonexistence of $u_{K'}$ $\Longrightarrow$ MP holds for $M \setminus K''$ by the last paragraph. Let $u \in \mathcal{H}(M \setminus K)$ with $u \le 1$, $\overline{\lim}_{p \to K} u(p) \le 0$. We want to show $u \le 0$ ($\Rightarrow$ MP for $M \setminus K$). First, by MP for $M \setminus K''$,

$$u|_{M \setminus K''} \le \max_{\partial K''} u.$$

Now, suppose $\max_{\partial K''} u > 0$. By the usual max-min principle in $K'' \setminus K$, and $\overline{\lim}_{p \to K} u \le 0 \left(< \max_{\partial K''} u\right)$, we would have

$$u|_{K'' \setminus K} \le \max_{\partial K''} u.$$

By the 2 displayed stmts., u attains its maximum at a point of $\partial K''$, which is interior to $M \setminus K$ $\underset{\text{usual MP}}{\Longrightarrow}$ u constant. This is a contradiction, since it is > 0 somewhere on $\partial K''$ and $\to \le 0$ elsewhere. Hence, $\max_{\partial K''} u \le 0$ and by the usual maximum principle in $K'' \setminus K$, $u_{K'' \setminus K} \le 0$. Also $u_{M \setminus K''} \le 0$, and so $u \le 0$.

by validity of MP for $M \setminus K''$ $\square$

$\boxed{\text{Corollary 1}}$ <u>If it exists, $g(p,p_0)$ satisfies the 3 properties:</u>

(I) $g(p,p_0) > 0$

(II) $\inf g(p,p_0) = 0$

(III) $g(p,p_0) + \log|z(p)|$ has a harmonic extension to a nbhd. of p_0.

Proof: We already know (I). For (III), let $m(r) := \max_{|z(p)|=r} g(p,p_0)$.

$(**) \Longrightarrow m(r) + \log(r)$ is an increasing function of r

$\Longrightarrow g(p,p_0) + \log|z(p)|$ is bounded <u>above</u> near p_0.

Also, $v(p) = \begin{cases} -\log|z(p)| + \log r_0, & |z(p)| < r_0 \\ 0, & \text{otherwise} \end{cases} \in V_{p_0}$

$\Longrightarrow g(p,p_0) \geq -\log|z(p)| + \log r_0$

$\Longrightarrow g(p,p_0) + \log|z(p)|$ bounded <u>below</u> near p_0.

Since isolated singularities of a bounded harmonic fcn. are removable, done.

For (II), set $c = \inf g(p,p_0)$. By (III), $g(p,p_0) + \log|z(p)|$ has a finite limit as $p \to p_0$. So (by max-min)

$$\underbrace{(1-\epsilon)v(p)}_{\text{growth bdd. as } p\to p_0 \text{ by } -(1-\epsilon)\log|z(p)|;\ \text{also, } 0 \text{ outside compact set}} \leq g(p,p_0) - c \qquad (\forall\, v \in V_{p_0},\ \epsilon > 0)$$

$\Longrightarrow (1-\epsilon)\, g(p,p_0) \leq g(p,p_0) - c \qquad (\forall\, \epsilon > 0)$

(defn. of g)

$\Longrightarrow c \leq 0 \Longrightarrow c = 0.$

(defn of c; fact that $g > 0$)

$\square$

$\boxed{\text{Corollary 2}}$ If $\exists$ bounded nonconstant $h \in \mathcal{H}(M)$, then $\forall$ compact $K \subset M$ w/ nonempty interior, the maximum principle on $M \setminus K$ fails, and M has a Green's function w/ singularity at any point $p_0 \in M$.

Proof: h has a maximum on K, attained (necessarily) on ∂K. Were MP for $M \setminus K$ valid, it would follow that this was a maximum on all of M, rendering h constant. $\square$

Now recall the Dirichlet principle. The proof generalises immediately to Riemann surfaces in the following sense:

$\boxed{\text{Proposition 3}}$ Let $M \subset \tilde{M}$ be RSs. If $\cancel{\emptyset} \neq \partial M \subset \tilde{M}$ is a finite union of closed analytic arcs, then $\forall$ bounded C^0 fcns. $f : \partial M \to \mathbb{R}$, $\exists$ $h : \overline{M} \to \mathbb{R}$ s.t.

- $|h| \leq \sup |f|$
- $h|_M \in \mathcal{H}(M)$
- $h|_{\partial M} = f$.

$\boxed{\text{Corollary 3}}$ Under the same hypothesis,

(i) M has Green's functions $g(p, p_0)$

(ii) $\lim\limits_{p \to \partial M} g(p, p_0) = 0$.

Proof: (i) is an immediate consequence of Prop. 3 + Cor. 2.

For (ii), let C be a small circle about p_0. Prop. 3 $\Rightarrow$ $\exists$ solution h to the Dirichlet problem outside C w/boundary data $h|_C = g$, $h|_{\partial M} = 0$. By the maximum principle for $\underline{\mathcal{H}}$, h = upper bound for all $v \in V_{p_0}$ $\Rightarrow$ $\overline{\lim_{p \to \partial M}}\, g(p,p_0) \leq 0$

$\Rightarrow$ $\underline{\lim}_{p \to \partial M}\, g(p,p_0) \geq 0$. $\square$

0-function $\in V_{p_0}$