Lecture 35: The Uniformization Theorem

Let M be a Riemann surface. Recall that Green's functions $g(p, p_0)$ exist [resp. don't exist] all points $p_0 \in M$ ($\Leftrightarrow$ harmonic measures exist [resp. don't exist] for all compact sets $K \subset M$). We will say "M has Green's functions".

Definition If M is noncompact with Green's functions, we call M *hyperbolic*. (e.g. $M = \mathbb{D}$ (or $\mathbb{H}$)) If M is noncompact w/o Green's functions, we call M *parabolic*. (e.g. $M = \mathbb{C}$)

We will prove our main result in the following form:

Uniformization Theorem Assume that M is simply connected. Then M is conformally isomorphic to $\mathbb{D}$ (if hyperbolic), the plane $\mathbb{C}$ (if parabolic), or the Riemann sphere $\hat{\mathbb{C}}$ (if compact).

The proof is broken into 3 sections, one for each of these cases.

I. Hyperbolic case

Let $M' = M \setminus \{p_0\}$, and choose a simply connected nbhd. $N_p \subset M'$ about each $p \in M'$, so that $g := g(-, p_0)$ has a well-defined harmonic conjugate there; call this h_p. Set

$$f_p := e^{-(g + ih_p)} \in \mathcal{H}ol(N_p) \,;$$

this is unique up to multiplication by an $e^{i\theta}$.

Now let $N_0 \subset M$ be a simply connected nbhd. about p_0, with local coordinate z. We know that $g(p, p_0) + \log|z(p)|$ belongs to $\mathcal{H}(N_0)$, so it has a well-defined harmonic conjugate h_0 there; and we can put

$$f_0 := z \cdot e^{-(g + \log|z| + ih_0)} \in \mathcal{H}ol(N_0) .$$

This vanishes to first order at p_0.

Given any curve γ from p_0 to p, by adjusting $e^{i\theta}$'s the f_p's are analytic continuations of f_0. Since M is simply connected, the monodromy theorem tells us that they patch together to give a global analytic function $f: M \to \mathbb{C}$.

Next, $|f(p)| = e^{-g(p,p_0)} < 1$ since $g(p,p_0) > 0$. It suffices to prove f is 1-to-1, for then one can get an isomorphism of M onto a bounded plane region, whereupon the Riemann mapping theorem produces an isomorphism $M \cong D$. Write $f(p,p_0)$ for f.

Consider (for some $p_1 \in M \setminus \{p_0\}$)

$$F(p) := \frac{f(p,p_0) - f(p_1,p_0)}{1 - \underbrace{\overline{f(p_1,p_0)}}\,\underbrace{f(p,p_0)}} \in \mathcal{H}ol(M)$$

(both have $|\cdot| < 1$)

$$= \varphi_{f(p_1,p_0)}(f(p,p_0)),$$

with $F(p_1) = 0$ and $|F| < 1$ (why?). If z_1 is a local coordinate at p_1, and $v \in V_{p_1}$ (in the notation of Lectures 33-34), then $\epsilon > 0 \Rightarrow \overline{\lim\limits_{p \to p_1}} \left[v(p) + (1+\epsilon)\log|z_1(p)| \right] = -\infty$

$$\Rightarrow \overline{\lim_{p \to p_1}} \left[v(p) + (1+\epsilon)\log|F(p)| \right] = -\infty .$$

By the maximum principle for $\underline{\mathcal{H}}(M)$, together with $|F| < 1$ and $\lim\limits_{p \to \partial M} v(p) = 0$, $v(p) + (1+\epsilon)\log|F(p)| \le 0$ on M.

Taking $\epsilon \to 0$ and $v \to g$, we have

$$(\dagger) \qquad \boxed{g(p,p_1) + \log|F(p)| \le 0}$$

hence $|F(p)| \le |f(p,p_1)|$. For $p = p_0$, this gives

$$|f(p_1,p_0)| \underset{\substack{\Uparrow \\ (f(p_0,p_0)=0)}}{=} |F(p_0)| \le |f(p_0,p_1)|.$$

But the argument is symmetric in p_1 & p_0, so we get the reverse inequality hence

$$|f(p_1,p_0)| = |f(p_0,p_1)|.$$

Conclude that (†) is an equality for $p = p_0$, i.e. $g(p_0,p_1) + \log|F(p_0)| = 0$. But then LHS (†) attains its maximum at $p = p_0$. Since LHS (†) $\in \mathcal{H}(M \setminus p_1)$, LHS(†) $\equiv 0$ by the maximum principle

$$\Longrightarrow |F(p)| = |f(p,p_1)|$$

$$\Longrightarrow F(p) = e^{i\theta} f(p,p_1) \text{ w/constant } \theta \in \mathbb{R}$$

$$\Longrightarrow F(p) \text{ has only zero at } p = p_1$$

($f(p,p_1)$ is only zero at $p = p_1$, & to 1st order)

$$\Longrightarrow f(p,p_0) = f(p_1,p_0) \text{ only at } p = p_1$$

(defn. of F)

$$\Longrightarrow f(p,p_0) \text{ is 1-to-1},$$

as desired. (So we have shown M hyperbolic $\Rightarrow M \cong \mathbb{D}$.)

<u>Remark</u> // From the fact that $g \to 0$ on ∂M, $f \to 1$ there, so in fact f already maps M <u>onto</u> $\mathbb{D}$, giving the isomorphism w/o any use of the RMT. //

II. Parabolic case

Henceforth we assume M is simply connected, without Green's functions.

Definition A divergent curve on M is a piecewise analytic simple arc $\varphi : [0,\infty) \to M$ s.t. $\varphi^{-1}(K)$ is compact $\forall\ K \subset M$ compact.

Assume M admits one, and set $M_t := M \setminus \varphi([t,\infty))$.

Lemma A: $\forall t \geq 0$

- M_t is simply connected
- $\forall p_0 \in M_0$, M_t admits GF w/ singularity at p_0
- $\lim\limits_{p \to \partial M_t} g(p, p_0) = 0$.

Proof: The latter two bullets follow at once from Cor. 3 of Lecture 34. To check the first, let $\gamma \subset M_{t_0}$ be a loop based at p_0, and set $\mathcal{A}(\gamma) := \{ t \in [t_0,\infty) \mid \alpha \underset{\substack{\uparrow \\ \text{homotopic to}}}{\sim} \{p_0\} \text{ in } M_t \}$. Now $\gamma \sim \{p_0\}$ on M, and the homotopy has compact image, so $\mathcal{A}(\gamma)$ is nonempty & open in $[t_0,\infty)$. If $t_1 \in \mathcal{A}(\gamma)$, then $[t_1,\infty) \subset \mathcal{A}(\gamma)$; set $t_2 := \inf\{ t \mid t \in \mathcal{A}(\gamma)\}$, so that $\gamma \sim \{p_0\}$ in $M_{t_2+\epsilon}$.

Consider a disk $\Delta \subset M \setminus \gamma$ about $\varphi(t_2)$ also containing $\varphi(t_2+\epsilon)$. There is a C^∞ automorphism of the closed disk which is the identity on the boundary and maps $\varphi([t_2+\varepsilon,\infty)) \cap \Delta$ onto $\varphi([t_2,\infty)) \cap \Delta$. Since the homotopy doesn't meet the former, its composition with this automorphism (on Δ) won't meet the latter. Thus $\gamma \sim \{p_0\}$ in M_{t_2} $\Longrightarrow$ $A(\gamma) \ni t_2$ $\Longrightarrow$ $A(\gamma)$ is closed $\Longrightarrow$ $A(\gamma) = [t_0, \infty)$ $\Longrightarrow$ $\gamma \sim \{p_0\}$ in M_{t_0}. $\square$

Next we recall the following corollary of Köbe distortion ($\Rightarrow$ local boundedness of schlicht functions) + Montel :

Lemma B : $\mathcal{S}_r := \{ f \in \mathcal{H}ol(D_r) \mid f(0)=0,\ f'(0)=1,\ f \text{ 1-to-1} \}$ is compact in the normal topology.

By lemma A, M_t is hyperbolic ; so by §I there exists a holomorphic isomorphism $f_t : M_t \xrightarrow{\cong} D$ for each t.

We may assume $f_t(p_0) = 0$ for some fixed $p_0 \in M$.

Choose $t_\ell \to \infty$, and denote M_{t_ℓ}, f_{t_ℓ} by M_ℓ, f_ℓ.

Fix a local coordinate $z = f_0(p)$ near p_0, and set

$$c_\ell := f_\ell'(z(p))\Big|_{p=p_0},$$

$$F_\ell := \frac{f_\ell}{c_\ell} : M_\ell \xrightarrow{\cong} D_{1/c_\ell} =: D_\ell .$$

Write $N_1 = \mathbb{Z}_{>0}$, and (inductively) given $N_\ell \subset \mathbb{Z}_{>0}$:

$\rotatebox{90}{$\in$}$

$j \geq \ell \ (\Rightarrow M_j \supseteq M_\ell)$

$F_j'(z(p_0)) = 1$, F_j defined/injective on M_ℓ

$\Rightarrow F_j \circ F_\ell^{-1} : D_\ell \hookrightarrow \mathbb{C}$ sends $0 \mapsto 0$, has derivative $= 1$ at 0

by Lemma B

$\Rightarrow \exists\, N_{\ell+1} \subset N_\ell$ s.t. $\{F_j \circ F_\ell^{-1}\}_{j \in N_{\ell+1}}$ converges to

$H_\ell : D_\ell \hookrightarrow \mathbb{C}$ (sending $0 \mapsto 0$, with derivative $= 1$ at 0).

(δ_{1/c_ℓ})

Let $n_j :=$ j^{th} entry in N_j ("diagonal subsequence"). Then

$$k > \ell \implies (H_k \circ F_k)\big|_{M_\ell} \text{ injective \& holomorphic}$$

$\shortparallel$ ← defn. of H_k

$$\left(\lim_{j\to\infty} F_{n_j} \circ F_k^{-1}\right) \circ F_k = \left(\lim_{j\to\infty} F_{n_j} \circ \underbrace{F_\ell^{-1}\right) \circ F_\ell} \circ F_k^{-1} \circ F_k$$

$$= \left(\lim_{j\to\infty} F_{n_j} \circ F_\ell^{-1}\right) \circ F_\ell$$

$$= H_\ell \circ F_\ell$$

$\Rightarrow H_k \circ F_k =$ extension of $H_\ell \circ F_\ell$ to M_k

$\Rightarrow \exists\, f : M \hookrightarrow \mathbb{C}$ extending all the $H_k \circ F_k$.

k arbitrary

Since M is simply connected, and f is 1-to-1 ($\Rightarrow$ homeomorphism onto its image), $f(M)$ is simply connected.

Suppose $f(M) \neq \mathbb{C}$. By the Riemann Mapping Theorem, we have

$$M \underbrace{\xrightarrow{\cong} f(M) \xrightarrow{\cong}}_{=:h} D,$$

and clearly $\mathrm{Re}(h)$ is a bounded, non constant harmonic function. By Cor. 2 of Lecture 34, M has Green's functions, a contradiction. So $(M \cong)\ f(M) \cong \mathbb{C}$.

But the proof is incomplete without

Lemma C: All parabolic RSs have a divergent curve.

Sublemma: M simply connected RS w/o divergent curve $\Longrightarrow$
$M\setminus\{p\}$ simply connected for any p.

Proof: As a topological space, M is a connected orientable 2-manifold. These are all homeomorphic to a sphere with $g \geq 0$ handles and $n \geq 0$ points removed. If $(g,n) \neq (0,0)$ or $(0,1)$, then π_1 is nontrivial. ⨳ So M is homeomorphic to $\mathbb{C}$ or $\hat{\mathbb{C}}$. But $\mathbb{C}$ has the divergent curve $[0,\infty)$, and thus M is homeo. to $\hat{\mathbb{C}}$; and $\hat{\mathbb{C}} \setminus \text{point} \underset{\text{homeo.}}{\cong} \mathbb{C}$ is indeed simply connected. □

Proof of Lemma C: Given M parabolic w/o divergent curves, $M\setminus\{p\}$ remains simply connected, and clearly has divergent curves (approaching p).

If $M\setminus\{p\}$ is hyperbolic, then $\exists f : M\setminus\{p\} \xrightarrow{\cong} D$ (bounded)
$\underset{\text{removable sing.}}{\xRightarrow{\text{Riemann}}} \exists \tilde{f} : M \to \bar{D}$ extending f $\underset{+ f \text{ nonconst.}}{\xRightarrow{\text{max princ.}}} |\tilde{f}(p)| < 1$
$\Longrightarrow \tilde{f} : M \to D$ (clearly NOT 1-to-1, with $\alpha = \tilde{f}(p) = \tilde{f}(q)$)
$\underset{\text{OMT}}{\Longrightarrow}$ $\exists$ disks about α in image of disks about p, q
$\Longrightarrow$ f itself not 1-to-1. ⨳

So $M \setminus \{p\}$ must be parabolic. Since it has divergent curves, $M \setminus \{p\} \cong \mathbb{C}$ (conformal iso.). Moreover, M parabolic $\Longrightarrow$ noncompact $\Longrightarrow$ M homeomorphic (topological iso.) to $\mathbb{C}$ $\Longrightarrow$ $M \setminus \{p\}$ homeomorphic to $\mathbb{C}^*$ ⨳. □

So M parabolic $\Longrightarrow$ $M \cong \mathbb{C}$ (conformal iso.).

III. The compact case

Let M be a simply-connected, compact R.S.

$M \setminus \{p\}$ noncompact $\Longrightarrow$ $M \setminus \{p\} \cong \mathbb{C}$ or $\mathbb{D}$, by §§ I & II.

If it's $\mathbb{D}$, Riemann removable sing. $\Longrightarrow$ extends to $M \to \overline{\mathbb{D}}$

$\Longrightarrow$ $M \setminus \{p\} \to \mathbb{D}$ not 1-1. ⨳

So $M \setminus \{p\} \xrightarrow[F]{\cong} \mathbb{C}$, and M is the (unique) 1-point compactification of $\mathbb{C}$, namely $\hat{\mathbb{C}}$.