Lecture 36: Several complex variables I

Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain (or more generally an open set).

$\boxed{\text{Definition}}$ A function $f : \mathcal{D} \to \mathbb{C}$ is holomorphic $\Leftrightarrow$ $\forall\, \underline{\alpha} \in \mathcal{D}$ $\exists$ open $U_{\underline{\alpha}} \subset \mathcal{D}$ s.t. f has a power series expansion $f(\underline{z}) = \sum_{\underline{i} \in \mathbb{Z}^n_{\geq 0}} c_{\underline{i}} (\underline{z} - \underline{\alpha})^{\underline{i}}$ which converges $\forall\, \underline{z} \in U$.

i.e. $(z_1, \ldots, z_n)$

i.e. $(z_1 - \alpha_1)^{i_1} \times \cdots \times (z_n - \alpha_n)^{i_n}$

In this lecture, which begins the final segment of the course, we shall only cover a few preliminaries, the most important of which is Osgood's lemma. (Finishing up Uniformization took some class time.)

Notation: $\overleftrightarrow{D}(\underline{\alpha}, \underline{r}) := \overleftrightarrow{D}(\alpha_1, r_1) \times \cdots \times \overleftrightarrow{D}(\alpha_n, r_n)$ polydisks

Clearly $f \in \mathcal{H}ol(\mathcal{D}) \Rightarrow$ expansion converges absolutely/uniformly on each U in the definition

Uniform limit of C^0 functions is C^0

$\Longrightarrow$ (i) $f \in C^0(\mathcal{D})$, in particular f is locally bounded (bounded on each compact subset of $\mathcal{D}$)

(ii) f is holomorphic in each variable separately (since it is a limit, fixing all but one variable, of holomorphic polynomials),

i.e. $\bar{\partial} f := \underbrace{\frac{\partial f}{\partial \bar{z}_1}}_{} d\bar{z}_1 + \cdots + \underbrace{\frac{\partial f}{\partial \bar{z}_n}}_{} d\bar{z}_n = 0$.

each of these vanishes

In fact, the converse holds:

$\boxed{\text{Lemma 1 (Osgood)}}$ f locally bounded, with $\underbrace{\bar{\partial} f = 0}$

$\Longrightarrow$ f holomorphic.

(i.e. f is holo. in each entry with the other entries fixed)

Proof: Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathcal{D}$,

$\bar{\Delta} := \bar{D}(\underline{\alpha}, \underline{r}) \subset \mathcal{D}$. Fix $(z_1, \dots, z_n) \in \Delta$.

f holo. in ζ_n $\Longrightarrow$ $f(\zeta_1, \dots, \zeta_{n-1}, z_n) = \frac{1}{2\pi i} \oint_{|\zeta_n - \alpha_n| = r_n} \frac{f(\zeta_1, \dots, \zeta_{n-1}, \zeta_n)\, d\zeta_n}{\zeta_n - z_n}$.

f holo in ζ_{n-1} $\Longrightarrow$ $f(\zeta_1, \dots, \zeta_{n-2}, z_{n-1}, z_n) = \frac{1}{2\pi i} \oint_{|\zeta_{n-1} - \alpha_{n-1}| = r_{n-1}} \frac{f(\zeta_1, \dots, \zeta_{n-1}, z_n)\, d\zeta_{n-1}}{\zeta_{n-1} - z_{n-1}}$

$$= \frac{1}{(2\pi i)^2} \oint_{|\zeta_{n-1} - \alpha_{n-1}| = r_{n-1}} \left(\oint_{|\zeta_n - \alpha_n| = r_n} \frac{f(\zeta_1, \dots, \zeta_n)\, d\zeta_n}{\zeta_n - z_n} \right) \frac{d\zeta_{n-1}}{\zeta_{n-1} - z_{n-1}},$$

etc.

Local boundedness allows us to rearrange the resulting iterated integral to obtain

$$(1) \qquad f(z_1,\dots,z_n) = \frac{1}{(2\pi i)^n} \int_{\substack{n \\ \times \\ i=1}\{|s_i-\alpha_i|=r_i\}} \frac{f(s_1,\dots,s_n)\prod ds_i}{\prod (s_i - z_i)} .$$

Now

$$\frac{1}{s_i - z_i} = \frac{1}{(s_i-\alpha_i)-(z_i-\alpha_i)} = \frac{1}{s_i-\alpha_i}\cdot\frac{1}{1-\frac{z_i-\alpha_i}{s_i-\alpha_i}} = \sum_{j\geq 0}\frac{(z_i-\alpha_i)^j}{(s_i-\alpha_i)^{j+1}}$$

$$\Rightarrow f(\underline{z}) = \sum_{\underline{j}\in \mathbb{Z}_{\geq 0}^{\times n}} \underbrace{\left(\frac{1}{(2\pi i)^n}\int \frac{f(s_1,\dots,s_n)\prod_{i=1}^n ds_i}{\prod_{i=1}^n (s_i-\alpha_i)^{j_i+1}}\right)}_{=:\gamma_{j_1,\dots,j_n} =: \gamma_{\underline{j}}} \underbrace{\prod (z_i-\alpha_i)^{j_i}}_{\text{or}: (\underline{z}-\underline{\alpha})^{\underline{j}}}$$

gives a power-series expansion at $\underline{\alpha}$, necessarily convergent in Δ. □

Corollary 1 (Cauchy estimates) Let $\underline{a} = (a_1,\dots,a_n)$ be a multi-index, and write $|\underline{a}| = \sum a_i$, $\underline{a}! = \prod a_i!$. In the above situation we have

$$\underline{a}!\,\gamma_{\underline{a}} = \left\{\left(\frac{\partial}{\partial \underline{z}}\right)^{\underline{a}} f\right\}(\underline{\alpha}),$$

with $|\gamma_{\underline{a}}| \leq M/\underline{r}^{\underline{a}}$, where $M = \sup_{\underline{z}\in\Delta} |f(\underline{z})|$.

Proof: Differentiating both sides of (1) gives

$$\left\{\left(\frac{\partial}{\partial \underline{z}}\right)^{\underline{a}} f\right\}(\underline{z}) = \frac{\underline{a}!}{(2\pi i)^n}\int \frac{f(\underline{s})\prod ds_i}{\prod (s_i - z_i)^{a_i+1}} .$$

Setting $\underline{z} = \underline{\alpha}$ leads immediately to both results. □

$\boxed{\text{Corollary 2 (Jensen's inequality)}}$ $\log|f(\underline{\alpha})| \le \frac{1}{\mathrm{vol}(\Delta)}\int_\Delta \log|f(\underline{z})|\, \underbrace{dV(\underline{z})}_{\prod r_i\, dr_i\, d\theta_i}$

Proof: Iterating the 1-variable inequality

$$\log|g(0)| \le \frac{1}{2\pi}\int_0^{2\pi}\log|g(\rho e^{i\theta})|\,d\theta$$

gives

$$\log|f(\underline{\alpha})| \le \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|f(\underline{\alpha}+\underline{\rho}\, e^{i\underline{\theta}})|\prod d\theta_j .$$

Take the product with $\prod \rho_j$, then $\int_0^{r_1}\cdots\int_0^{r_n} d\rho_1\cdots d\rho_n$ to obtain

$$\frac{1}{\cancel{2^n}}\prod r_i^2 \log|f(\underline{\alpha})| \le \frac{1}{\cancel{2^n}\pi^n}\int \log|f(\underline{z})|\,dV(\underline{z})$$

hence the result. □