Lecture 37: Several complex variables II

Recall that on an open set $\mathcal{D} \subset \mathbb{C}^n$

$$f \in \mathcal{H}ol(\mathcal{D}) \overset{\text{def}}{\iff} f \text{ has (convergent) local power series developments about each point}$$

$$\overset{\text{Osgood}}{\iff} f \text{ is bounded on compact sets and holo. in each variable separately (i.e. } \bar{\partial} f = 0)$$

If the local power series development is $\sum \gamma_{\underline{n}} (\underline{z} - \underline{\alpha})^{\underline{n}}$, convergent on $\bar{\Delta} = \bar{D}(\underline{\alpha}, \underline{r})$, we had

$$\underline{n}!\, \gamma_{\underline{n}} = \left(\frac{\partial}{\partial \underline{z}}\right)^{\underline{n}} f \Big|_{\underline{z} = \underline{\alpha}} \quad \Big(\underbrace{\text{with } |\gamma_{\underline{n}}| \leq \frac{\sup_\Delta |f|}{\underline{r}^{\underline{n}}}}_{\text{Cauchy estimates}}\Big)$$

and

$$\log |f(\underline{\alpha})| \leq \frac{1}{\text{vol}(\Delta)} \int_\Delta \log |f(\underline{z})| \, dV(\underline{z}) \qquad \text{(Jensen)}.$$

<u>Notation</u>:

$$\left.\begin{aligned} \overset{(-)}{D}(\underline{\alpha}, \underline{r}) &:= \overset{(-)}{D}(\alpha_1, r_1) \times \cdots \times \overset{(-)}{D}(\alpha_n, r_n) \\ \overset{(-)}{D}(\underline{\alpha}, r) &:= \overset{(-)}{D}(\alpha_1, r) \times \cdots \times \overset{(-)}{D}(\alpha_n, r). \end{aligned}\right\} \text{polydisks}$$

$$\left.\begin{aligned} \overset{(-)}{B}(\underline{\alpha}, r) &:= \{\underline{z} \mid |\underline{z} - \underline{\alpha}| < r\} \\ &= \{\underline{z} \mid \textstyle\sum (z_i - \alpha_i)^2 < r^2\} \end{aligned}\right\} \text{balls}$$

I. What's similar

(A) A family of normally convergent $f_\alpha \in \mathcal{H}ol(\mathcal{D})$ converges to a holomorphic function.

Proof: By the 1-var. result, this holds for all the 1-variable "slices" of f. Further, since each f_α is C^0, the limit will be C^0 ($\Rightarrow$ locally bounded). Done by Osgood. $\square$

(B) A holomorphic function which is $\equiv 0$ on a (nonempty) open subset is $\equiv 0$.

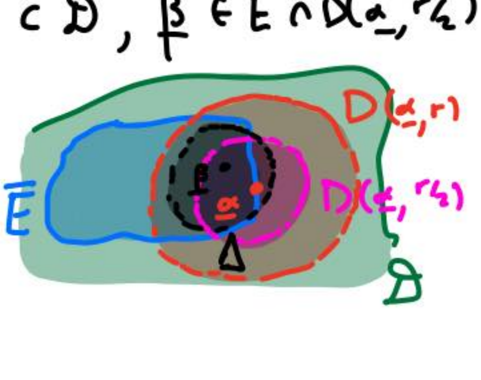
Proof: Let $f \in \mathcal{H}ol(\mathcal{D})$,

$$E := \operatorname{Int}\{\underline{z} \in \mathcal{D} \mid f(\underline{z}) = 0\} \text{ (nonempty \& open by assumption).}$$

If $E \subset \mathcal{D}$ relatively closed, then since $\mathcal{D}$ is connected, $E = \mathcal{D}$ and we are done.

Let $\underline{\alpha} \in \mathcal{D} \cap \overline{E}$, $D(\underline{\alpha}, r) \subset \mathcal{D}$, $\underline{\beta} \in E \cap D(\underline{\alpha}, r/2)$.

Then $\underline{\alpha} \in \underbrace{D(\underline{\beta}, r/2)}_{=:\Delta} \subset \mathcal{D}$.



We need to show $\Delta \subset E$.

Now $f \equiv 0$ on a possibly smaller neighborhood of $\underline{\beta}$.

Arguing with power-series coefficients OR by using holomorphicity in each variable to extend on slices now gets the job done. $\square$

(C) **Maximum modulus principle**: $f \in \mathcal{H}ol(\mathcal{D})$, $\underline{\alpha} \in \mathcal{D}$, $|f(\underline{\alpha})| \geq |f(\underline{z})| \ \forall \underline{z} \in \mathcal{D} \implies f$ constant.

Proof: $\overline{\Delta} := \overline{D}(\underline{\alpha}, \underline{r}) \subset \mathcal{D}$. By Jensen,

$$\mathrm{vol}(\Delta)\,|f(\underline{\alpha})| \leq \int_\Delta |f(\underline{z})|\,dV.$$

If $|f(\underline{\alpha})| \geq |f(\underline{z})| \ \forall z \in \Delta$, then

$$0 \leq \int_\Delta \big(|f(\underline{\alpha})| - |f(\underline{z})|\big)\,dV = \mathrm{vol}(\Delta)|f(\underline{\alpha})| - \int_\Delta |f(\underline{z})|\,dV \leq 0$$

$\Rightarrow |f(\underline{\alpha})| - |f(\underline{z})| = 0 \ \forall \underline{z} \in \Delta$

$\Rightarrow f(\underline{z})$ has constant modulus in all z_i-directions from all points of Δ

$\Rightarrow f(\underline{z})$ is constant in all z_i-directions from all points of Δ

1-variable theorem

$\Rightarrow f$ is constant, hence $\equiv f(\underline{\alpha})$ on Δ

$\Rightarrow f \equiv f(\underline{\alpha})$ on $\mathcal{D}$. $\square$

(B)

II. What's different

Ⓐ The power series extends to any polydisk about $\underline{\alpha}$ contained in $\mathcal{D}$. But these polydisks are not nested. From this we see that the domains on which power series are defined are not all of one form.

Examples //
- $\sum (z_1+z_2)^j$ converges on $|z_1+z_2|<1$
- $\sum (z_1 z_2)^j$ converges on $|z_1 z_2|<1$
- $\sum z_1^j$ converges on $|z_1|<1$ //

Ⓑ Nothing like the RMT or uniformization is true.

This is best exemplified by the

$\boxed{\text{Theorem 1 (Poincaré)}}$ $D_1^n := D(0,1)^{\times n}$ and $B_1^n := B(\underline{0},1)$ are not biholomorphic for $n \geq 2$.

First of all, note that we are saying there does NOT exist a 1-1 / onto map

$$(z_1, \dots, z_n) \longmapsto (f_1(z_1,\dots,z_n), \dots, f_n(z_1,\dots,z_n))$$

$$\overset{\cap}{D_1^n} \qquad\qquad \overset{\cap}{B_1^n}$$

where the f_i are in $\mathcal{H}ol(D_1^n)$.

Next, set

$$U(n) := \left\{ g \in GL(n,\mathbb{C}) \,\middle|\, {}^t g \bar{g} = id_n \right\}.$$

Given $\underline{z} \in B_1^n$ $(\Leftrightarrow {}^t\underline{z} \cdot \bar{\underline{z}} < 1)$, we have for $g \in U(n)$

view as column vector $\in \mathbb{C}^n$

$${}^t(g\underline{z}) \cdot \overline{g\underline{z}} = {}^t\underline{z}\, \underbrace{{}^t g \bar{g}}_{id.}\, \bar{\underline{z}} = {}^t\underline{z} \cdot \bar{\underline{z}} < 1 \implies g\underline{z} \in B_1^n$$

(and $g^{-1} \in U(n)$)

$\Rightarrow \quad g \in Aut_{\underline{0}}(B_1^n) := \{$holomorphic automorphisms fixing the origin $\underline{0}\}$.

Now, $U(n)$ is connected, and <u>nonabelian</u> for $n \geq 2$.

<u>Claim</u>: (a) There exists an automorphism of D_1^n sending any $\underline{\alpha} \in D_1^n$ to $\underline{0}$.

(b) The identity connected component of $Aut_{\underline{0}}(D_1^n)$ is <u>abelian</u>.

Were there to be a biholomorphism $D_1^n \xrightarrow[\phi]{\cong} B_1^n$ sending $0 \mapsto 0$, $Aut_{\underline{0}}(B_1^n)$ would be $\phi \circ Aut_{\underline{0}}(D_1^n) \circ \phi^{-1}$ — clearly impossible if Claim (b) holds. (Further, if there were a biholo $D_1^n \xrightarrow{\cong} B_1^n$
$\underline{\alpha} \mapsto \underline{0}$

then composing it with an automorphism as in Claim (a) reduces us to sending $\underline{0} \mapsto \underline{0}$.) So it only remains to prove the Claim, for which we need 2 lemmas.

Lemma A: $\Omega \subset \mathbb{C}^n$ bounded domain, $\Phi : \Omega \to \Omega$ fixing $\underline{\alpha}$, $d\Phi_{\underline{\alpha}} = id$. $\Longrightarrow \Phi = id_{\Omega}$.

Proof: wlog $\underline{\alpha} = \underline{0}$. Assume $\Phi(\underline{z}) \neq \underline{z}$.

$$\Phi(\underline{z}) = \underline{z} + \underline{P}_k(\underline{z}) + O(|\underline{z}|^{k+1})$$

(where $\underline{P}_k$ = 1st nonzero homogeneous polynomial in the Taylor expansion)

$$\Rightarrow (\Phi \circ \Phi)(\underline{z}) = \underline{z} + 2\,\underline{P}_k(\underline{z}) + O(|\underline{z}|^{k+1})$$

$$\Rightarrow (\underbrace{\Phi \circ \cdots \circ \Phi}_{j})(\underline{z}) = \underline{z} + j\,\underline{P}_k(\underline{z}) + O(|\underline{z}|^{k+1}).$$

Now, given r, R s.t. $D^n_r \subset \underset{\text{domain}(\Phi\circ\cdots\circ\Phi)}{\Omega} \subset D^n_R$, the Cauchy estimates

$\Longrightarrow$ for each component of $\Phi^j = \Phi \circ \cdots \circ \Phi$, and $|\underline{a}| = k$,

$$\left| \left\{ \left(\frac{\partial}{\partial \underline{z}} \right)^{\underline{a}} \Phi^j_{\underset{\{1,\dots,n\}}{\ell}} \right\} (\underline{0}) \right| \leq \frac{R \cdot \underline{a}!}{r^k}$$

bounds the maximum

$$\Longrightarrow \left| \left\{ \left(\frac{\partial}{\partial \underline{z}} \right)^{\underline{a}} \Phi^j \right\} (\underline{0}) \right| \leq n \cdot \frac{R \cdot \underline{a}!}{r^k}$$

$$\parallel$$

$$j \left| \left\{ \left(\frac{\partial}{\partial \underline{z}} \right)^{\underline{a}} \Phi \right\} (\underline{0}) \right|$$

$$\underset{j\to\infty}{\Longrightarrow} \quad \left\{\left(\frac{\partial}{\partial \underline{z}}\right)^{\underline{a}} \Phi\right\}(\underline{0}) = \underline{0}$$

$$\Longrightarrow \quad \underline{P}_k = 0 \quad \times\!\!\times \qquad \square$$

<u>Lemma B</u>: Given $\underline{0} \in \Omega \subset \mathbb{C}^n$ bounded circular domain, i.e. $\underline{z} \in \Omega \Rightarrow e^{i\underline{\theta}}\underline{z} \in \Omega$ $(\forall\, \underline{\theta} \in \mathbb{R}^n)$, let $\Phi \in \mathrm{Aut}(\Omega)$ fix $\underline{0}$. Then Φ is linear.

<u>Proof</u>: $\Phi' := \mathrm{Jac}_{\underline{0}}\Phi$; $\rho_\theta \in \mathrm{Aut}(\Omega)$ is $\rho_\theta(z) := (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$.

Set $\Psi := \rho_{-\theta} \circ \Phi^{-1} \circ \rho_\theta \circ \Phi$. Then

$$\Psi' = \rho_{-\theta}' \circ (\Phi^{-1})' \circ \rho_\theta' \circ \Phi' \quad (\rho_{\pm\theta}' \text{ are diagonal \& commute w/ everything})$$

$$= (\Phi^{-1})' \circ \Phi' = \mathrm{id}.$$

$$\overset{\text{Lemma A}}{\Longrightarrow} \quad \Psi(\underline{z}) \equiv \underline{z}$$

$$\Longrightarrow \quad \Phi \circ \rho_\theta = \rho_\theta \circ \Phi \quad (\forall\, \theta)$$

$$\Longrightarrow \quad \Phi \text{ linear (think scaling!)}. \qquad \square$$

<u>Proof of Claim: (a)</u> $\Phi \in \mathrm{Aut}(D_1^n)$, $\Phi(\underline{0}) = \underline{\alpha}$ given.

$$\Psi(\underline{z}) := \left(\frac{z_1 - \alpha_1}{1 - \bar{\alpha}_1 z_1}, \ldots, \frac{z_n - \alpha_n}{1 - \bar{\alpha}_n z_n}\right) \quad \rightsquigarrow$$

$$f := \Psi \circ \Phi \in \mathrm{Aut}_{\underline{0}}(D_1^n) \quad (\text{sends } \underline{0} \mapsto \underline{0}).$$

(b) By Lemma B, f is linear.

$\Rightarrow f(\underline{z}) = [b_{ij}]\underline{z}$ (matrix acting on column vector).

Set $z^{ik} := \left(\left(1-\frac{1}{k}\right)\frac{\overline{b_{i1}}}{|b_{i1}|}, \ldots, \left(1-\frac{1}{k}\right)\frac{\overline{b_{in}}}{|b_{in}|}\right) \in D_1^n$

Then the ith component of $f(z^{ik})$ is $\sum_{j=1}^{n}\left(1-\frac{1}{k}\right)|b_{ij}| < 1$

(since $f \in \mathrm{Aut}_{\underline{0}}(D_1^n) \Rightarrow f(z^{ik}) \in D_1^n$)

$\underset{k\to\infty}{\Longrightarrow} \sum_j |b_{ij}| \le 1.$

Also, $w^{jk} := (0, \ldots, 0, 1-\frac{1}{k}, 0, \ldots, 0) \underset{k\to\infty}{\longrightarrow} \partial D^n$ (outside compacta)

$\Rightarrow f(w^{jk}) = \left(\left(1-\frac{1}{k}\right)b_{1j}, \ldots, \left(1-\frac{1}{k}\right)b_{nj}\right) \longrightarrow \partial D^n$ (")

$f \in \mathrm{Aut}_{\underline{0}}(D_1^n)$

$\Rightarrow \max_{1\le i\le n} |b_{ij}| = 1.$

$\Rightarrow$ each row of $[|b|]$ sums to ≤ 1; each column of $[b]$ has an entry of modulus 1

$\Rightarrow [|b|]$ is a permutation matrix

$\Rightarrow [b]$ is a permutation composed with an $e^{i\underline{\theta}}$

$\Rightarrow$ identity connected component of $\mathrm{Aut}_{\underline{0}}(D_1^n)$ is all $e^{i\underline{\theta}}$'s : transformations of form $(z_1, \ldots, z_n) \mapsto (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$

$\Rightarrow \mathrm{Aut}_{\underline{0}}(D_1^n)^{0}$ is abelian.

id. component

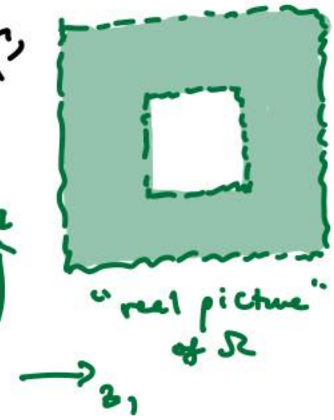
$\square$

(C) Hartogs phenomenon

Let $\Omega = D_3^2 \setminus \overline{D_1^2} \subset \mathbb{C}^2$, $f \in \mathrm{Hol}(\Omega)$.

Then without any boundedness assumption, f has a holomorphic extension to D_3^2.

(We will formalize this later.)



Proof: Fix $|z_1| < 3$.

Then $f(z_1, z_2)$ viewed as a function of z_2 is a holomorphic function in the annulus $1 < |z_2| < 3$, hence has a Laurent expansion

$$f(z_1, z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1)\, z_2^j .$$

We have

$$(*) \qquad a_j(z_1) = \frac{1}{2\pi i} \int_{|\zeta| = 2} \frac{f(z_1, \zeta)}{\zeta^{j+1}}\, d\zeta$$

which depends holomorphically on z_1 since

$$\oint_{\substack{\text{small} \\ \text{circle } \gamma}} a_j(z_1)\, dz_1 = \frac{1}{2\pi i} \int_{|\zeta|=2} \frac{1}{\zeta^{j+1}} \Big(\underbrace{\oint f(z_1, \zeta)\, dz_1}_{\substack{= 0 \text{ by analyticity of } f \text{ in} \\ \text{the } z_1\text{-direction}}} \Big) d\zeta$$

$\Longrightarrow$ $a_j(\cdot)$ holo. (use Morera).

But by (*), $a_j(z_1) = 0$ for $j < 0$ and $1 < |z_1| < 3$,

since then f is a holo. function of z_2 on $\underline{\underline{D_3}}$.

By the Theorem §I A above in 1 variable,

$a_j(z_1) \equiv 0$ for $j < 0$. But then, the original

series expansion becomes

$$\sum_{j \geq 0} a_j(z_1) z_2^j ,$$

defining a holomorphic function on D_3^2. $\square$