Lecture 39 : Several complex variables IV

I. Hartogs phenomenon (bis)

- $D \subset \mathbb{C}^n$ bounded domain, $n > 1$
- $K \subset D$ compact s.t. $D \setminus K$ connected

Theorem 1 $f \in \mathcal{H}ol(D\setminus K) \Longrightarrow \exists\, F \in \mathcal{H}ol(D)$ with $F|_{D\setminus K} = f$.

Proof: $\exists\, \varphi \in C_c^\infty(D)$, $\varphi \equiv 1$ on $\underset{\text{(open)}}{\tilde{K}} \supset K$.

Define $\tilde{f}(\underline{z}) := \left\{ \begin{matrix} 0 & , \underline{z} \in K \\ (1-\varphi(\underline{z}))f(\underline{z}) & , \underline{z} \in D\setminus K \end{matrix} \right\} \in C^\infty(D)$. Then

$\psi(\underline{z}) := \bar{\partial}\tilde{f}(\underline{z}) \in \mathcal{E}^{0,1}(C_c^\infty(D))$ is compactly supported (say, on $K_0 \supset K$) because φ is, and where $\varphi \equiv 0$, $\tilde{f} \equiv f \Rightarrow \bar{\partial}\tilde{f} = \bar{\partial} f = 0$. We also have $\bar{\partial}\psi = \bar{\partial}^2 \tilde{f} = 0$.

By $\underset{\text{(Thm. 2)}}{\text{Lecture}}$ 38, $\exists\, u \in C_c^\infty(D)$ s.t. $\bar{\partial} u = \psi$; b/c it is compactly supported, $u \equiv 0$ on (wolog) $U = D \setminus K_0$.

Set $F := \tilde{f} - u \implies \bar{\partial} F = \bar{\partial}\tilde{f} - \bar{\partial} u = \psi - \psi = 0$

$\implies F \in \mathcal{H}ol(D)$.

$F|_U = (\tilde{f} - u)|_U = \tilde{f}|_U = f|_U \Rightarrow F, f$ agree near ∂D

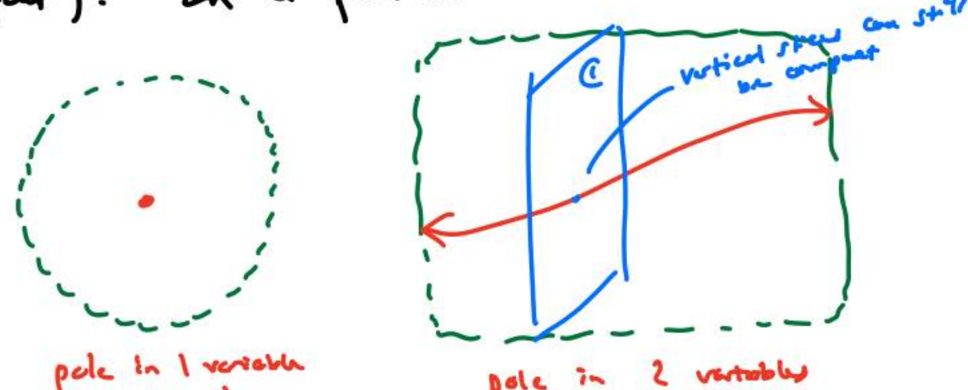
$\Rightarrow$ agree on any conn. open to which they both have analytic continuation: namely, $\mathcal{D} \setminus K$. (Lect 37 & I.B) $\square$

<u>Remark</u> // At first glance, one wonders if the Hartogs phenomenon says something stupid, like: if $\mathcal{D} = \mathbb{C}^2$ and $K = \mathbb{C} \times \{0\}$, then $\mathcal{D} \setminus K$ is connected; and given $f(w) \in \mathcal{H}ol(\mathbb{C} \setminus \{0\})$ you can define on $\mathcal{D} \setminus K$

$F(z,w) := f(w) \underset{??}{\xrightarrow{\text{Hartogs}}}$ extends to $\mathcal{D}$, contradicting the possibility of f having a pole at 0.

This isn't a problem: <u>$\mathbb{C} \times \{0\}$ is not compact</u>.

Also, functions of several variables can have poles too. But the basic idea is: these poles cannot be along a compact subset of the domain where they are defined; nor can they occur "in complex codimension ≥ 2" (since such a subset would have ≥ 2-variable "slices" which are compact). In a picture:



pole in 1 variable (compact)

pole in 2 variables (noncompact — goes all the way to the boundary!) //

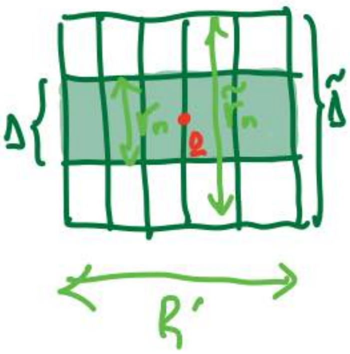
II. Hartogs theorem

We now prove that the ingredient "f locally bounded" in Osgood (f loc. bdd. + $\bar\partial f = 0 \Rightarrow f$ holo.) is unnecessary.

<u>Hartogs Lemma</u>: Given: $\Delta = D(\underline{0}, \underline{R}) = D(\underline{0}', \underline{R}') \times D(0, r_n) \subset \mathbb{C}^n$;

$f \in \mathcal{H}ol(\Delta)$, $f_\nu \in \mathcal{H}ol(D(\underline{0}', \underline{R}'))$;

$$(*) \quad f(\underline{z}) = \sum_\nu f_\nu(\underline{z}') z_n^\nu \text{ on } \Delta.$$



Suppose $\exists\, \tilde{r}_n > r_n$ s.t. $\forall\, \underline{z}' \in D(\underline{0}', \underline{R}')$, $(*)$ converges in $D(0, \tilde{r}_n)$. Then <u>$(*)$ converges uniformly on any compact subset of $\tilde{\Delta} := D(\underline{0}', \underline{R}') \times D(0, \tilde{r}_n)$, extending f to $\mathcal{H}ol(\tilde{\Delta})$</u>.

Proof: We must show that $(*)$ converges uniformly on $K \subset \tilde{\Delta}$, since then we have a normal limit of holomorphic functions.

STEP 1 Let $\underline{\alpha}' \in D(\underline{0}', R')$, $\overline{D}(\underline{\alpha}', \underline{s}') \subset D(\underline{0}', \underline{R}')$, $0 < s_n < r_n < \tilde{s}_n < \tilde{r}_n$. Let $M \in [1, \infty)$ be an upper bound for $|f|\big|_{\overline{D}(\underline{\alpha}', \underline{s}') \times \overline{D}(0, s_n)} \ (\Subset \Delta)$.

Cauchy inequalities $\Longrightarrow$ $|f_\nu(\underline{z}')|\, s_n^\nu \leq M$ $(\forall\, \underline{z}' \in D(\underline{\alpha}', \underline{s}'), \nu \in \mathbb{N})$

$$\Longrightarrow \boxed{\tfrac{1}{\nu} \log |f_\nu(\underline{z}')| \leq \tfrac{1}{\nu} \log M - \log s_n \leq \log M - \log s_n \leq M_0 \quad (\forall\, \underline{z}' \in D(\underline{\alpha}', \underline{s}'), \nu \in \mathbb{N})} \quad (\#)$$

for some constant.

STEP 2 Let $\underline{z}' \in \overline{D}(\underline{\alpha}', \underline{s}')$ be fixed. Then $(*)$ converges in $D(0, \tilde{r}_n)$.

$$\xRightarrow{\text{Cauchy}} \lim_{\nu \to \infty} |f_\nu(\underline{z}')|\, u_n^\nu = 0 \quad \text{for } u_n \in (\tilde{s}_n, \tilde{r}_n)$$

$$\Longrightarrow \boxed{\overline{\lim_{\nu \to \infty}} \frac{1}{\nu} \log |f_\nu(\underline{z}')| \le -\log u_n \quad \left(\forall \underline{z}' \in \overline{D}(\underline{\alpha}', \underline{s}')\right).} \quad (\#\#)$$

For slightly smaller $\overline{D}(\underline{\alpha}'; \underline{t}') \subset D(\underline{\alpha}', \underline{s}')$, $(\#) \Longrightarrow$ the functions $\frac{1}{\nu} \log |f_\nu(\underline{t}')|$ are <u>uniformly</u> bounded in $D(\underline{\alpha}'; \underline{t}')$ (by M_0). Now

$$(\#\#) \xRightarrow{\text{Fatou}} \overline{\lim_{\nu}} \int_{\overline{D}(\underline{\alpha}'; \underline{t}')} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV \le \int_{\overline{D}(\underline{\alpha}'; \underline{t}')} \overline{\lim_{\nu}} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV$$

$$\le -\left\{ \mathrm{vol}\left(D(\underline{\alpha}'; \underline{t}')\right) \right\} \log(u_n)$$

$$\xRightarrow{t_n \in (\tilde{s}_n, u_n)} \int_{\overline{D}(\underline{\alpha}'; \underline{t}')} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV \le -\left\{ \mathrm{vol}\left(D(\underline{\alpha}'; \underline{t}')\right) \right\} \log(t_n)$$

for $\nu \gg 0$ (say, $\ge \nu_0$)

$$\Longrightarrow \int_{D(\underline{w}'; \underline{t}' + \epsilon)} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV = \int_{D(\underline{\alpha}', \underline{t}') \cup \{D(\underline{w}'; \underline{t}' + \epsilon) \setminus D(\underline{\alpha}'; \underline{t}')\}} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV$$

$\overline{D}(\underline{\alpha}'; \underline{t}') \subseteq D(\underline{\alpha}', \underline{s}')$

$\Downarrow$ $(\forall \underline{w}' \in D(\underline{\alpha}'; \epsilon))$

$\overline{D}(\underline{\alpha}'; \underline{t}') \subseteq D(\underline{w}'; \underline{t}' + \epsilon) \subset D(\underline{\alpha}'; \underline{s}')$

for $\epsilon > 0$ suff. small

$$\le -\left\{ \mathrm{vol}\left(D(\underline{\alpha}'; \underline{t}')\right) \right\} \log t_n + M_0 \int_{D(\underline{w}'; \underline{t}' + \epsilon) \setminus D(\underline{\alpha}'; \underline{t}')} dV$$

$$\underset{(\epsilon > 0 \text{ suff. small})}{\le} -\left\{ \mathrm{vol}\left(D(\underline{w}'; \underline{t}' + \epsilon)\right) \right\} \log \tilde{s}_n.$$

$$\xRightarrow{\text{Jensen}} \frac{1}{\nu} \log |f_\nu(\underline{w}')| \le \frac{1}{\mathrm{vol}\left(D(\underline{w}'; \underline{t}' + \epsilon)\right)} \int_{D(\underline{w}'; \underline{t}' + \epsilon)} \frac{1}{\nu} \log |f_\nu(\underline{z}')|\, dV$$

$$\le -\log \tilde{s}_n$$

$$\Longrightarrow |f_\nu(\underline{w}')|\, \tilde{s}_n^\nu \le 1 \quad \forall\, \underline{w}' \in D(\underline{\alpha}', \epsilon)$$

$$\Longrightarrow \text{series } (*) \text{ AC \& UC in } D(\underline{\alpha}', \epsilon) \times D(0, \tilde{s}_n)$$

$\Longrightarrow$ Series (*) AC & UC in any compact subset of $D(\underline{0}, \underline{R}') \times D(0, \tilde{r}_n)$. $\square$

$\underline{\alpha} \in D(\underline{0}, \underline{R}')$
$\tilde{s}_n < \tilde{r}_n$ arbitrary

$\boxed{\text{Theorem 2 (Hartogs)}}$ Let $f : \mathcal{D} \to \mathbb{C}$ be a complex-valued function, which is separately holomorphic in each variable. Then $f \in \mathcal{H}ol(\mathcal{D})$.

Proof: Induce on dimension. Trivial for dim. 1.

Assume for 1 thru $n-1$. Let $f \in \widetilde{\mathcal{H}ol}(\mathcal{D}) :=$ $\{\mathbb{C}\text{-valued fcns. with } \bar{\partial} f = 0\}$. We only really need to show f bounded on compact subsets, or equivalently on all $\bar{D}(\underline{\alpha}', \underline{R}') \times \bar{D}(\alpha_n, R_n) = \bar{D}(\underline{\alpha}, \underline{R}) \subset \mathcal{D}$. (In fact what we'll do is just to show that f is holo. at an arbitrary $\underline{\alpha} \in \mathcal{D}$.)

$\boxed{\text{STEP 1}}$ $\left(\exists \begin{cases} \beta_n \in D(\alpha_n, R_n/2) \\ \delta > 0 \end{cases} \text{s.t.} \begin{cases} f \text{ is bounded on} \\ D(\underline{\alpha}', \underline{R}') \times D(\beta_n, \delta) \subset D(\underline{\alpha}, \underline{R}) \end{cases} \right)$

Pf: Let $X_m := \left\{ z_n \in \bar{D}(\alpha_n, R_n/2) \,\middle|\, |f(\underline{z}', z_n)| \leq m \ \forall \underline{z}' \in \bar{D}(\underline{\alpha}', \underline{R}') \right\}$.

For any fixed z_n, $f(\underline{z}', z_n)$ is holo. in $\underline{z}'$ by the inductive hypothesis, hence $|f|$ bounded on $\bar{D}(\underline{\alpha}', \underline{R}') \Rightarrow \cup X_m = \bar{D}(\alpha_n, R_n/2)$.

Baire category thm. $\Rightarrow$ $\mathrm{int}(X_m)$ not all empty

$\Rightarrow X_{m_0} \supset \Delta(\beta_n, \delta)$ as above. $\square$

$\boxed{\text{STEP 2}}$ Step 1 $\xRightarrow{\text{Osgood}}$ f holo. on $D(\underline{\alpha}', \underline{R}') \times D(\beta_n, \delta)$.

Let $s_n > \frac{R_n}{2}$ s.t. $D(\beta_n, s_n) \subset D(\alpha_n, R_n)$.

For each fixed $\underline{z}' \in D(\underline{\alpha}', \underline{R}')$, $f(\underline{z}', z_n)$ is holo. in z_n

in disk $D(\beta_n, s_n)$ $\xRightarrow[\text{Hartogs lemma}]{}$ f holo. in

$D(\underline{\alpha}', \underline{R}') \times D(\beta_n, s_n) \ni \underline{\alpha}$. $\square$

III. Square-integrable functions

We conclude by laying some groundwork for Friday's lecture. First we recall some standard concepts from operator theory/functional analysis.

$\boxed{\text{Definition}}$ A Hilbert space is a $\mathbb{C}$-vector space with inner-product $\langle \cdot , \cdot \rangle$ linear in the first argument, such that $\langle y, x \rangle = \overline{\langle x, y \rangle}$ and $\langle x, x \rangle \geq 0$ (with equality $\Leftrightarrow$ $x = 0$), which is complete with respect to the metric defined by $\|x\| := \langle x, x \rangle^{1/2}$. One says that a Hilbert space is separable $\Leftrightarrow$ it admits a countable orthonormal basis.

$\boxed{\text{Riesz Representation Theorem}}$ Let H be a Hilbert space, H^* the dual Hilbert space of continuous linear functionals (with norm $\|\varphi\| = \sup_{\|x\|=1} |\varphi(x)|$), and define a map $\varphi : H \to H^*$ by $\varphi(x) := \varphi_x$, where $\varphi_x(y) := \langle y, x \rangle$. Then φ is an isometry; in particular, it is surjective.

Now let $\mathcal{D} \subset \mathbb{C}^n$ be a domain. We will take for granted that $L^2(\mathcal{D})$ is a separable Hilbert space. We consider the space of square-integrable holomorphic functions

$$A^2(\mathcal{D}) := \left\{ f \in \mathcal{H}ol(\mathcal{D}) \,\middle|\, \underbrace{\left(\int_{\mathcal{D}} |f(\underline{z})|^2 \, dV\right)^{1/2}}_{=: \|f\|_{A^2(\mathcal{D})}} < \infty \right\}$$

with inner product

$$\langle f, g \rangle := \int_{\mathcal{D}} f(\underline{z}) \overline{g(\underline{z})} \, dV(\underline{z}) \quad , \quad f, g \in A^2(\mathcal{D}).$$

Proposition With this inner product, $A^2(\mathcal{D})$ is a Hilbert space.

Proof: Clear except for completeness. Given $\{f_j\} \subseteq A^2(\mathcal{D})$, Cauchy in $\|\cdot\|_{A^2(\mathcal{D})}$, $f_j \to f \in L^2(\mathcal{D})$ by completeness of L^2. We want to show that $f \in \mathcal{H}ol(\mathcal{D})$.

Let $K \subset \mathcal{D}$ be compact, $r_k := d(K, \partial \mathcal{D})$.

By Jensen, for $\underline{z}^\circ \in \mathrm{int}(K)$

$$\log|f(\underline{z}^\circ)| \leq \mathrm{vol}(B(\underline{z}^\circ, r_k))^{-1} \int_{B(\underline{z}^\circ, r_k)} \log|f(\underline{z})| \, dV$$

(convexity of log) $\leadsto$ $\leq \log\left(\dfrac{\int_{B(\underline{z}^\circ, r)} |f(\underline{z})| \, dV}{\mathrm{vol}(B(\underline{z}^\circ, r_k))}\right)$

$$\Longrightarrow \quad |f(\underline{z}^\circ)| \leq \mathrm{vol}(B(\underline{z}^\circ, r_k))^{-1} \int_{B(\underline{z}^\circ, r_k)} |f(\underline{z})| \, dV$$

$\leftarrow (|f(\underline{z})|^2)^{1/2}$

(convexity of $\sqrt{\ }$) $\leadsto$ $\leq \mathrm{vol}(B(\underline{z}^\circ, r_k))^{-1/2} \|f\|_{A^2(\mathcal{D})}$

$$\leq \frac{C(n)}{r_K^n} \|f\|_{A^2(\Omega)} =: C_K \|f\|_{A^2(\Omega)}$$

$$\Longrightarrow \underbrace{\|f\|_K}^{\text{sup norm}} \leq C_K \|f\|_{A^2(\Omega)} \quad (*)$$

$\Longrightarrow$ {norm convergence $\Rightarrow$ uniform convergence on K}

$\Longrightarrow$ f holo. on $\mathrm{int}(K)$, done (as K was arbitrary). $\square$